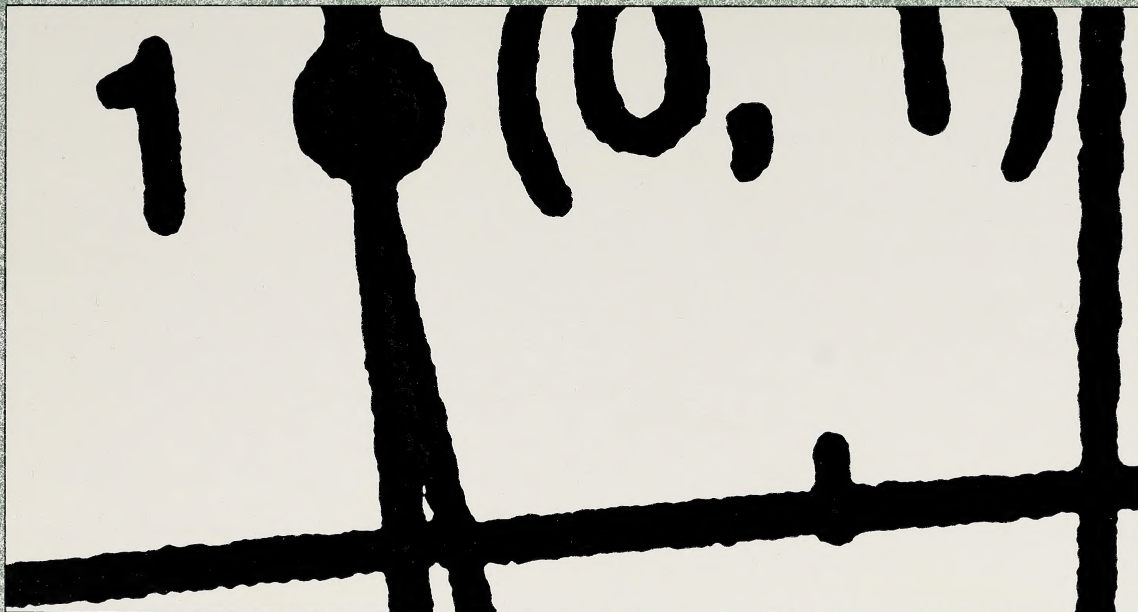
MATHEMATICS 31



Distance Learning

UNIT 2: DIFFERENTIATION OF ALGEBRAIC EXPRESSIONS AND GRAPHING

Alberta EDUCATION

Welcome

Distance Learning

You have chosen an alternate form of learning that allows you to work at your own pace. You will be responsible for your own schedule, for disciplining yourself to study the units thoroughly, and for completing your units regularly. We wish you much success and enjoyment in your studies.

Mathematics 31 Student Module Unit 2 Differentiation of Algebraic Expressions and Graphing Alberta Distance Learning Centre ISBN No. 0-7741-0291-8

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General Information

This information explains the basic layout of each booklet.

- **What You Already Know** and **Review** are to help you look back at what you have previously studied. The questions are to jog your memory and to prepare you for the learning that is going to happen in this unit.

- As you begin each **Topic**, spend a little time looking over the components. Doing this will give you a preview of what will be covered in the topic and will set your mind in the direction of learning.

- **Exploring the Topic** includes the objectives, concept development, and activities for each objective. Use your own papers to arrive at the answers in the activities.

- **Extra Help** reviews the topic. If you had any difficulty with **Exploring the Topic**, you may find this part helpful.

- **Extensions** gives you the opportunity to take the topic one step further.

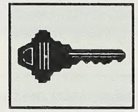
- To summarize what you have learned, and to find instructions on doing the unit assignment, turn to the **Unit Summary** at the end of the unit.

- The **Appendices** include the solutions to **Activities (Appendix A)** and any other charts, tables, etc. which may be referred to in the topics **(Appendix B, etc.)**.

Visual Cues

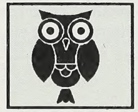
Visual cues are pictures that are used to identify important areas of the material. They are found throughout the booklet.

An explanation of what they mean is written beside each visual cue.

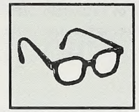


Key Idea
- flagging important ideas

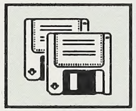
Audiotape
- learning by listening to an audiotape



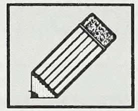
What You Already Know
- reviewing what you already know



Another View
- exploring different perspectives

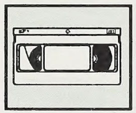


Computer Software
- learning by using computer software



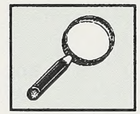
Review
- studying previous concepts

Solutions
- correcting the activities

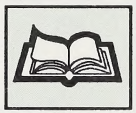


Videotape
- learning by viewing a videotape

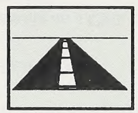
Introduction
- introducing the unit



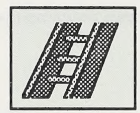
Extra Help
- providing additional study



Print Pathway
- choosing a print alternative



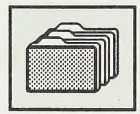
What Lies Ahead
- previewing the unit



Extensions
- going on with the topic

Calculator
- using your calculator

Exploring the Topic
- actively learning new concepts



What You Have Learned
- summarizing what you have learned

Mathematics 31

Course Overview

Mathematics 31 contains 9 units. Beside each unit is a percentage that indicates what the unit is worth in relation to the rest of the course. The units and their percentages are listed below. You will be studying the unit that is shaded.

Unit 1 Introduction to Differential Calculus	10%
Unit 2 Differentiation of Algebraic Expressions and Graphing	10%
Unit 3 Practical Application of Derivatives	20%
Unit 4 Integration	10%
Unit 5 Geometric Vectors and their Application	10%
Unit 6 Algebraic Vectors and their Application	10%
Unit 7 Inner Product	10%
Unit 8 Systems of Linear Equations	10%
Unit 9 Matrices and Linear Transformations	10%
	100%

Unit Assessment

After completing the unit you will be given a mark based totally on a unit assignment. This assignment will be found in the Assignment Booklet.

Unit Assignment - 100%

If you are working on a CML terminal, your teacher will determine what this assessment will be. It may be

Unit Assignment - 50%
Supervised Unit Test - 50%

Introduction to Differentiation of Algebraic Expressions and Graphing

This unit covers topics dealing with differentiation of algebraic expressions and graphing. Each topic contains explanations, examples, and activities to assist you in understanding differentiation of algebraic expressions and graphing. If you find you are having difficulty with the explanations and the way the material is presented, there is a section called **Extra Help**. If you would like to extend your knowledge of the topic, there is a section called **Extensions**.

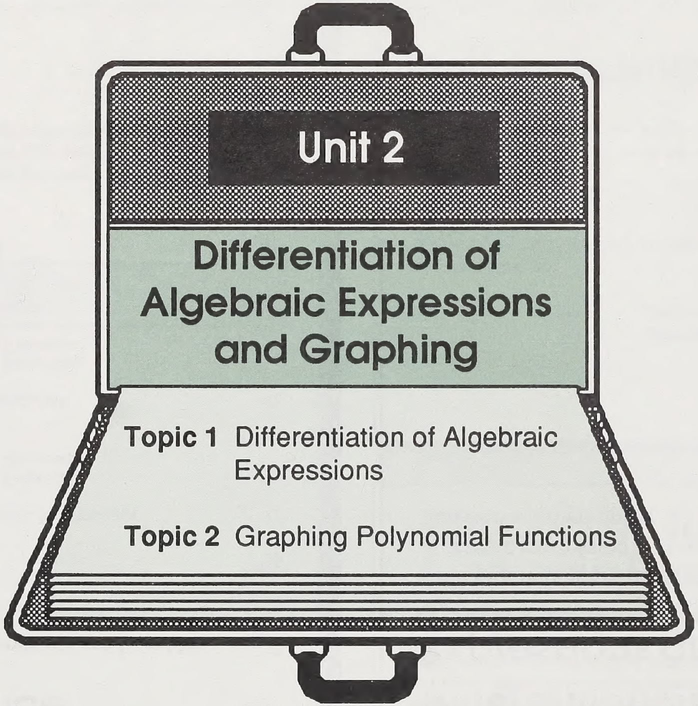
You can evaluate your understanding of each topic by working through the activities. Answers are found in the solutions in **Appendix A**. In several cases there is more than one way to do the question.

Unit 2 Differentiation of Algebraic Expressions and Graphing

Contents at a Glance

Differentiation of Algebraic Expressions and Graphing

This unit will help you develop rules for finding derivatives of algebraic expressions. It will also show you how to use derivatives in graphing.


Unit 2
Differentiation of Algebraic Expressions and Graphing
Topic 1 Differentiation of Algebraic Expressions
Topic 2 Graphing Polynomial Functions

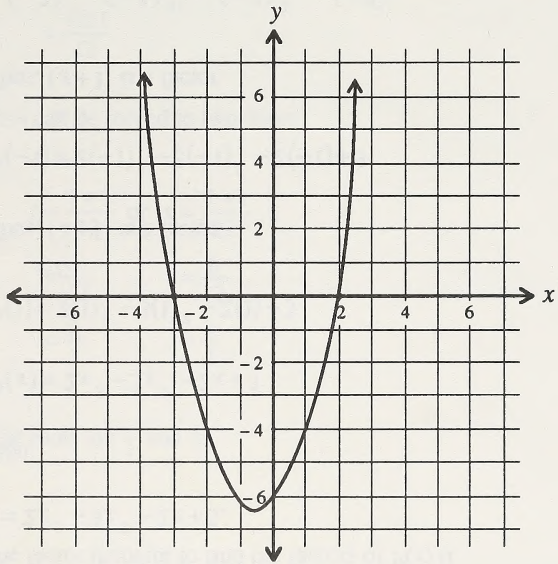
What You Already Know

Refresh your memory! Recall the following concepts:

- graphing of polynomial functions using the zeros and the y-intercept

  If $f(x)=x^2+x-6$, then $f(x)=-6$ when $x=0$.

  Also, if $f(x)=x^2+x-6$ or $f(x)=(x-2)(x+3)$, then the zeros are $+2$ and -3.



- factoring polynomials using traditional factoring techniques including the division algorithm and the factor theorem

Example 1

Use the method of inspection to factor the expression $6x^2+x-2$.

Solution:

$$6x^2+x-2=(2x-1)(3x+2)$$

Example 2

Use the division algorithm to find the other factor of x^3-8 if one factor is $(x-2)$.

Solution:

$$\begin{array}{r|l} & x^2+2x+4 \\ \hline x-2 & x^3+0x^2+0x-8 \\ & \underline{x^3-2x^2} \\ & \quad 2x^2+0x \\ & \quad \underline{2x^2-4x} \\ & \qquad 4x-8 \\ & \qquad \underline{4x-8} \\ & \qquad\quad 0 \end{array}$$

The factors of x^3-8 are $(x-2)$ and $\left(x^2+2x+4\right)$.

Example 3

Use the factor theorem to find the factors of $P(x)$ if $P(x)=2x^3-3x^2-2x+3$.

Solution:

$$P(x)=2x^3-3x^2-2x+3$$

$$P(1)=2(1)^3-3(1)^2-2(1)+3$$
$$=0$$

Thus, $(x-1)$ is a factor.

$$P(-1)=2(-1)^3-3(-1)^2-2(-1)+3$$
$$=0$$

Thus, $(x+1)$ is a factor.

$$P\left(\frac{-3}{2}\right)=2\left(\frac{-3}{2}\right)^3-3\left(\frac{-3}{2}\right)^2-2\left(\frac{-3}{2}\right)+3$$
$$=\frac{-30}{4}$$
$$=-7\tfrac{1}{2}$$

Thus, $(2x+3)$ is not a factor.

$$P\left(\frac{3}{2}\right)=2\left(\frac{3}{2}\right)^3-3\left(\frac{3}{2}\right)^2-2\left(\frac{3}{2}\right)+3$$
$$=0$$

Thus, $(2x-3)$ is a factor.

Therefore, $P(x)=(x-1)(x+1)(2x-3)$.

- solving quadratic equations by factoring and using the quadratic formula

Example 4

Solve the quadratic equation $6x^2-7x+2=0$ using the method of factoring.

Solution:

$$6x^2-7x+2=0$$
$$\left(6x^2-3x-4x+2\right)=0$$
$$3x(2x-1)-2(2x-1)=0$$
$$(3x-2)(2x-1)=0$$

(This is called the method of decomposition.)

Therefore, if $3x-2=0$, then $x=\frac{2}{3}$; if $2x-1=0$, then $x=\frac{1}{2}$.

The solution of the equation is $x=\frac{1}{2}$ and $x=\frac{2}{3}$.

Use the quadratic formula $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ to find the roots of the same equation.

First write $6x^2 - 7x + 2 = 0$ where $a = 6$, $b = -7$, and $c = 2$.
Then, substitute the values in the formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-7) \pm \sqrt{49 - 4(6)(2)}}{2(6)} \\ &= \frac{7 \pm \sqrt{49 - 48}}{12} \\ &= \frac{7 \pm \sqrt{1}}{12} \\ &= \frac{7 \pm 1}{12} \end{aligned}$$

This can be solved in two ways.

$$\begin{aligned} x &= \frac{7+1}{12} \quad \text{or} \quad & x &= \frac{7-1}{12} \\ &= \frac{8}{12} & &= \frac{6}{12} \\ &= \frac{2}{3} & &= \frac{1}{2} \end{aligned}$$

The roots are $\frac{1}{2}$ and $\frac{2}{3}$.

- finding the equations of lines given certain geometric conditions

Example 5

Find the equation of a line that intersects the y-axis at the point $(0, -6)$ and is parallel to the line $2y - x + 10 = 0$.

Solution:

First find the slope of the line $2y - x + 10 = 0$.

$$\begin{aligned} 2y &= x - 10 \\ y &= \frac{1}{2}x - 5 \end{aligned}$$

Therefore, the slope is $\frac{1}{2}$.

The line that passes through $(0, -6)$ has the slope of $\frac{1}{2}$ since it is parallel to $2y - x + 10 = 0$.

Recall the condition for parallelism is $m_1 = m_2$, where m is the slope of the line.

From $y = mx + b$, $m = \frac{1}{2}$ and $b = -6$.

Therefore, the equation is $y = \frac{1}{2}x - 6$ or $2y - x + 12 = 0$.

- relationships between the slopes of parallel lines and perpendicular lines

Example 6

Show if $2x+y+3=0$ is perpendicular to $2y-x-6=0$, that it is also perpendicular to $\frac{1}{2}x-y-7=0$.

Solution:

The condition for perpendicularity is $m_1 = \frac{-1}{m_2}$ or $m_1 m_2 = -1$, where m is the slope of the line.

Therefore, write $2x+y+3=0$ as $-y=+2x+3$ or $y=-2x-3$, where the slope $m_1 = -2$. Write $2y-x-6=0$ as $2y=x+6$ or $y=\frac{1}{2}x+3$, where the slope $m_2 = \frac{1}{2}$.

$$\therefore m_1 m_2 = (-2)\left(\frac{1}{2}\right) = -1$$

Since $\frac{1}{2}x-y-7=0$ can be written as $-y=\frac{-1}{2}x+7$ or $y=\frac{1}{2}x-7$, it has a slope of $\frac{1}{2}$.

$$\therefore m_1 m_2 = (-2)\left(\frac{1}{2}\right) = -1$$

Therefore, the line $\frac{1}{2}x-y-7=0$ is also perpendicular to the line $2x+y+3=0$.

Which of the following lines are parallel?

$$2x+y+3=0 \qquad 2y-x-6=0 \qquad \frac{1}{2}x-y-7=0$$

The condition for parallelism is $m_1 = m_2$.

Since $2y-x-6=0$ and $\frac{1}{2}x-y-7=0$ have the same slope $\frac{1}{2}$, then these two lines are parallel.

- solving systems of equations

Example 7

Find the values of x and y if $x^2+y^2=100$ and $xy=48$.

Solution:

$$x^2+y^2=100 \quad (1)$$
$$xy=48 \quad (2)$$

Multiply (2) by two and add to (1).

$$\begin{array}{rrl} & x^2+y^2=100 & (1) \\ 2\times(2): & +2xy=96 & (3) \\ \hline (1)+(3): & x^2+2xy+y^2=196 & \\ & (x+y)^2=196 & \\ & \therefore x+y=\pm 14 & (4) \end{array}$$

Multiply (2) by 2 and subtract from (1).

$$\begin{array}{rrl} & x^2 + y^2 = 100 & (1) \\ 2 \times (2): & 2xy = 96 & (3) \\ \hline (1) - (3): & x^2 - 2xy + y^2 = 4 & \\ & (x-y)^2 = 4 & \\ & x - y = \pm 2 & (5) \end{array}$$

From (4) and (5) the solutions are $(8, 6)$, $(6, 8)$, $(-8, -6)$, and $(-6, -8)$.

Note also that you could have done this by the method of substitution.

$$x^2 + y^2 = 100 \quad (1)$$

$$xy = 48$$

$$y = \frac{48}{x} \quad (2)$$

Substitute (2) in (1).

$$x^2 + \left(\frac{48}{x}\right)^2 = 100$$

$$x^4 - 100x^2 + 2304 = 0$$

Much more work is needed to solve this problem. As a result the first method is the preferred one in this case.

Review

Now that you have refreshed your memory, try the following questions to check your understanding.

1. Factor the following and state the zeros.

   a. $P(x)=2x^2-13x+6$

   b. $P(x)=x^4-x^3-3x^2+x+2$

2. Draw the graphs of the following. Graph paper is provided in **Appendix B**.

   a. $y=(x+2)^2(x-1)(x+1)$

   b. $P(x)=x^3-3x^2-x+3$

3. Solve the following by factoring or by using the quadratic formula.

   a. $x^2-x-20=0$

   b. $4x^2-8x=5$

   c. $3x^2-3-x=0$

4. Find the equation of a line with the following characteristics.

   a. contains $A(1,-6)$ with a slope of -3

   b. contains $A(-2,1)$ and $B(1,3)$

   c. has a slope of $\frac{1}{4}$ and y-intercept of 6

5. a. State the slope of a line perpendicular to the line $y=-4x+2$.

   b. State the slope of a line parallel to the line $2x+3y-6=0$.

6. Solve the following systems of equations.

   a. $2x+y-6=0$ ①
 $x-y+3=0$ ②

   b. $3x+2y+4=0$ ①
 $4x-3y+11=0$ ②

   c. $y=x^2+1$ ①
 $y=x+3$ ②

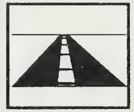
If you feel comfortable with these exercises, then you are ready to start the unit.

Now go to the **Review** solutions in **Appendix A**.

Topic 1 Differentiation of Algebraic Expressions

Introduction

In **Unit 1, Topic 2** the slope of the tangent to a curve is associated with the derivative of the function. You can use the derivative to find the slope of the tangent at a particular point. Must every derivative be developed using first principles? How do you find the derivative of functions other than polynomial functions? This topic will look at rules for finding derivatives of functions and the application of these rules.



What Lies Ahead

Throughout the topic you will learn to

1. determine the derivative of the sum or difference of two functions
2. use the power rule and its applications
3. use the product rule and its applications
4. use the chain rule and its applications
5. use the quotient rule and its applications
6. determine the derivatives of relations

Now that you know what to expect, turn the page to begin your study of differentiation of algebraic expressions.

Exploring Topic 1

Activity 1

Determine the derivative of the sum or difference of two functions.

In the first topic from first principles you found the derivative of a polynomial with more than one term. A polynomial with more than one term can easily be separated into two polynomials. Although the derivative of the sum or difference of two polynomials is equal to the sum or difference of the derivatives, they still require proof. You will find that the derivative theorems for the sum and difference of two functions are the corresponding theorems of the limit theorem for the sum or difference of two functions. Find the derivative of the sum or difference of two functions.

Let $f(x)$ and $g(x)$ be two functions.

Let $t(x) = f(x) \pm g(x)$.

You have to prove that $D_x t(x) = D_x \left[f(x) \pm g(x)\right]$.

If x changes by some small quantity Δx, then $f(x)$ becomes $f(x+\Delta x)$, $g(x)$ becomes $g(x+\Delta x)$, and $t(x)$ becomes $t(x+\Delta x)$.

Now substitute $x+\Delta x$ in the original equation.

$$t(x) = f(x) \pm g(x) \qquad (1)$$

$$t(x+\Delta x) = f(x+\Delta x) \pm g(x+\Delta x) \qquad (2)$$

$$(2)-(1):\quad t(x+\Delta x) - t(x) = \left[f(x+\Delta x) - f(x)\right] \pm \left[g(x+\Delta x) - g(x)\right]$$

Divide every term by Δx.

$$\frac{t(x+\Delta x)-t(x)}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x} \pm \frac{g(x+\Delta x)-g(x)}{\Delta x}$$

Take the limit of both sides as $\Delta x \to 0$.

$$\lim_{\Delta x \to 0} \frac{t(x+\Delta x)-t(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \pm \lim_{\Delta x \to 0} \frac{g(x+\Delta x)-g(x)}{\Delta x}$$

By definition,

this is $D_x t(x)$, this is $D_x f(x)$, and this is $D_x g(x)$.

Therefore, $D_x t(x) = D_x f(x) \pm D_x g(x)$.

This proves that the derivative of the sum or difference of two functions is equal to the sum or difference of the derivatives.

When $y = f(x)$, the derivative of y with respect to x can be represented by different notations including the following:

$\frac{dy}{dx}$ $\qquad y'$ $\qquad f'(x)$ $\qquad D_x y$

Now you can differentiate polynomials as shown in the following example.

Example 1

Differentiate $y=3x^4-2x^3+5x^2-x+7$.

Solution:

$$\begin{aligned}\frac{dy}{dx}&=\frac{d(3x^4)}{dx}-\frac{d(2x^3)}{dx}+\frac{d(5x^2)}{dx}-\frac{d(x)}{dx}+\frac{d(7)}{dx}\\&=12x^3-6x^2+10x-1\end{aligned}$$

Try the following questions. Do at least the odd-numbered questions.

Differentiate each of the following.

1. $y=2x^4-3x^3-5x+9$
2. $y=x^5-8x^4+3x^3$
3. $y=5x^4-\frac{1}{2}x^3+7$
4. $y=\frac{1}{2}x^3-2x^2+x-5$

For solutions to **Activity 1**, turn to **Appendix A, Topic 1.**

Activity 2

Use the power rule and its applications.

The derivatives of ax, ax^2, ax^3, and $a\left(\frac{1}{x}\right)$ were discussed in the previous unit. Now it is time to take another step forward. Find the derivative of x^n when n is any real natural number.

Consider $y=x^n$, where $n \in N$.
Assume when x changes by h, y changes by k; therefore,
$y+k=(x+h)^n$.

Now use the binomial theorem to expand this expression $(x+h)^n$.

$$\begin{aligned}y+k&=(x+h)^n\\&=x^n+nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\frac{n(n-1)(n-2)}{6}x^{n-3}h^3+\cdots+h^n\end{aligned}$$

Substitute $y=x^n$ in the preceding equation.

$$\begin{aligned}k&=nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\frac{n(n-1)(n-2)}{6}x^{n-3}h^3+\cdots+h^n\\&=h\left(nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}h+\frac{n(n-1)(n-2)}{6}x^{n-3}h^2+\cdots+h^{n-1}\right)\end{aligned}$$

$$\frac{k}{h}=\left(nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}h+\frac{n(n-1)(n-2)}{6}x^{n-3}h^2+\cdots+h^{n-1}\right)$$

Now find the limit when $h \to 0$.

$$\lim_{h\to 0}\left(\frac{k}{h}\right) = \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1}\right]$$
$$= \lim_{h\to 0} nx^{n-1} + \lim_{h\to 0}\frac{n(n-1)}{2}x^{n-2}h + \cdots + \lim_{h\to 0} h^{n-1}$$
$$= nx^{n-1} + 0 + \cdots + 0$$
$$= nx^{n-1}$$

Since h is the increase in x and k is the corresponding increase in y, $h = \Delta x$ and $k = \Delta y$.

$$\lim_{h\to 0}\frac{k}{h} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$
$$\therefore D_x y = nx^{n-1}$$

Since $y = x^n$, then $D_x\left(x^n\right) = nx^{n-1}$.

It can also be proved that $D_x\left(x^r\right) = rx^{r-1}$ where $x \in R$ and r is any rational number. If you want to see the proof, then look in **Extensions**. You may assume that $D_x\left(x^r\right) = rx^{r-1}$ where $x,\ r \in R$ is true without proof. Look at some examples.

Example 2

Find the derivative of x^5.

Solution:

Let $y = x^5$.

$$\frac{dy}{dx} = 5x^{5-1}$$
$$= 5x^4$$

Example 3

Find the derivative of $x^{\frac{1}{2}}$.

Solution:

Let $y = x^{\frac{1}{2}}$.

$$\frac{dy}{dx} = \left(\frac{1}{2}\right)x^{\frac{1}{2}-1}$$
$$= \frac{1}{2}x^{-\frac{1}{2}}$$
$$= \frac{1}{2x^{\frac{1}{2}}}$$

Example 4

Find the derivative of x^{-4}.

Solution:

Let $y = x^{-4}$.

$$\frac{dy}{dx} = (-4)x^{-4-1}$$
$$= -4x^{-5}$$
$$= \frac{-4}{x^5}$$

Example 5

Find the derivative of $x^3 + 2x^{-\frac{3}{2}} + 5$.

Solution:

Let $y = x^3 + 2x^{-\frac{3}{2}} + 5$.

$$D_x y = D_x\left(x^3\right) + D_x\left(2x^{-\frac{3}{2}}\right) + D_x(5)$$
$$= 3x^{3-1} + (2)\left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1} + 0$$
$$= 3x^2 - 3x^{-\frac{5}{2}}$$

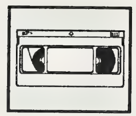
Example 6

Find the derivative of $\frac{3}{x^2} - 5x^{-\frac{1}{5}} + \frac{1}{2}x^{\frac{1}{2}}$.

Solution:

Let $y = \frac{3}{x^2} - 5x^{-\frac{1}{5}} + \frac{1}{2}x^{\frac{1}{2}}$.

$$y = 3x^{-2} - 5x^{-\frac{1}{5}} + \frac{1}{2}x^{\frac{1}{2}}$$
$$D_x y = D_x\left(3x^{-2}\right) - D_x\left(5x^{-\frac{1}{5}}\right) + D_x\left(\frac{1}{2}x^{\frac{1}{2}}\right)$$
$$= (-2)(3)x^{-3} - (5)\left(-\frac{1}{5}\right)x^{-\frac{1}{5}-1} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{\frac{1}{2}-1}$$
$$= -6x^{-3} + x^{-\frac{6}{5}} + \frac{1}{4}x^{-\frac{1}{2}}$$



For additional instruction on the derivative and the power rule, you may wish to view the video titled **The Derivative by First Principles, The Power Rule**. This video is program 2 of the *Catch 31*[1] series.

[1] *Catch 31* is a title of ACCESS Network.

Now it is your turn. Do the following exercises.

Find the derivatives of each of the following. (Do at least the odd-numbered questions.)

1. $y=3$
2. $y=5x^2$
3. $y=3x^4$
4. $y=1$
5. $y=8-5x+12x^2$
6. $y=7+6x^2+x^4$
7. $y=\frac{1}{2}x^5-6x^4+\frac{1}{3}x+3$
8. $y=\frac{1}{3}x^4-\frac{1}{2}x^3-5x+7$
9. $y=x^{\frac{1}{2}}-x^{\frac{5}{6}}$
10. $y=\frac{2}{3}x^{\frac{1}{3}}-2x^{\frac{1}{2}}$
11. $y=\frac{1}{5}x^{-\frac{1}{5}}-4x^{-\frac{1}{4}}$
12. $y=\frac{3}{x^{\frac{1}{3}}}+\frac{4}{3}x^{-\frac{1}{4}}$

For solutions to **Activity 2**, turn to **Appendix A, Topic 1**.

Activity 3

Use the product rule and its applications.

What is the derivative of $(x-1)(x+2)$? How do you find the derivative of the product of two functions? In this case you can find the product of $(x-1)$ and $(x+2)$ first, and then you can take the derivative of the product.

Let $y=(x-1)(x+2)$.

$$y=x^2+x-2$$
$$D_x y=2x+1$$

If the functions are not simple, this method can be a longer procedure than necessary. It is more convenient to find a general rule for the derivative of the product of two functions at x.

Let the two functions be $f(x)$ and $g(x)$.
If $y=f(x)\bullet g(x)$, find $D_x y$.

Let $m=f(x)$ and $n=g(x)$; then, $y=mn$.
If x changes to $x+\Delta x$, y changes to $y+\Delta y$.
Then, $f(x)$ changes to $f(x+\Delta x)$ (m changes to $m+\Delta m$) and $g(x)$ changes to $g(x+\Delta x)$ (n changes to $n+\Delta n$).

$$\therefore y+\Delta y=(m+\Delta m)(n+\Delta n)$$
$$=mn+m(\Delta n)+n(\Delta m)+(\Delta m)(\Delta n)$$

Since $y=mn$, then $\Delta y=m(\Delta n)+n(\Delta m)+(\Delta m)(\Delta n)$.

Divide both sides by Δx.

$$\frac{\Delta y}{\Delta x}=m\frac{\Delta n}{\Delta x}+n\frac{\Delta m}{\Delta x}+\frac{(\Delta m)(\Delta n)}{\Delta x}$$

Multiply the last term by $\frac{\Delta x}{\Delta x}$. $\left(\text{Note that } \frac{\Delta x}{\Delta x}=1.\right)$

$$\frac{\Delta y}{\Delta x}=m\frac{\Delta n}{\Delta x}+n\frac{\Delta m}{\Delta x}+\frac{\Delta m}{\Delta x}\frac{\Delta n}{\Delta x}(\Delta x)$$

Take the limit as $\Delta x\to 0$.

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0} m\frac{\Delta n}{\Delta x}+\lim_{\Delta x\to 0} n\frac{\Delta m}{\Delta x}+\lim_{\Delta x\to 0}\frac{\Delta m}{\Delta x}\frac{\Delta n}{\Delta x}(\Delta x)$$
$$=m\lim_{\Delta x\to 0}\frac{\Delta n}{\Delta x}+n\lim_{\Delta x\to 0}\frac{\Delta m}{\Delta x}+0$$
$$\frac{dy}{dx}=m\frac{dn}{dx}+n\frac{dm}{dx}$$

$$D_x(mn)=m\frac{dn}{dx}+n\frac{dm}{dx}$$

This is called the **product rule**.

Some books use u and v to denote $f(x)$ and $g(x)$.
Therefore, $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$. Thus, $D_x(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$.

Note that the derivative of a product is **not** equal to the product of the derivatives.

Now apply the product rule to find the derivative of a product.

Example 7

Find the derivative of $(x-1)(x+2)$.

Solution:

Let $y=(x-1)(x+2)$.

$$\frac{dy}{dx}=(x-1)\frac{d(x+2)}{dx}+(x+2)\frac{d(x-1)}{dx}$$
$$=(x-1)(1)+(x+2)(1)$$
$$=(x-1)+(x+2)$$
$$=2x+1$$

Example 8

Find the derivative of $\left(x^3+2x^2-3x+5\right)\left(x^2-x+8\right)$.

Solution:

Let $y=\left(x^3+2x^2-3x+5\right)\left(x^2-x+8\right)$.

$$
\begin{aligned}
\frac{dy}{dx} &= \left(x^3+2x^2-3x+5\right)\frac{d\left(x^2-x+8\right)}{dx}+\left(x^2-x+8\right)\frac{d\left(x^3+2x^2-3x+5\right)}{dx} \\
&= \left(x^3+2x^2-3x+5\right)(2x-1)+\left(x^2-x+8\right)\left(3x^2+4x-3\right) \\
&= 2x^4+4x^3-6x^2+10x-x^3-2x^2+3x-5+3x^4+4x^3-3x^2-3x^3-4x^2 \\
&\quad +3x+24x^2+32x-24 \\
&= 5x^4+4x^3+9x^2+48x-29
\end{aligned}
$$

Example 9

Find the derivative of $(x-3)^{-2}(x+1)$.

Solution:

Let $y=(x-3)^{-2}(x+1)$.

$$
\begin{aligned}
\frac{dy}{dx} &= (x-3)^{-2}\frac{d(x+1)}{dx}+(x+1)\frac{d(x-3)^{-2}}{dx} \\
&= (x-3)^{-2}(1)+(x+1)(-2)(x-3)^{-3}(1) \\
&= (x-3)^{-2}-2(x+1)(x-3)^{-3}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d(x-3)^{-2}}{dx} \\
&\quad =(-2)(x-3)^{-3}\frac{d(x-3)}{dx} \\
&\quad =-2(x-3)^{-3}(1) \\
&\quad =-2(x-3)^{-3}
\end{aligned}
$$

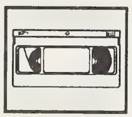
Example 10

Find the derivative of $(x-1)\sqrt{x+3}$.

Solution:

Let $y=(x-1)\sqrt{x+3}$.

$$\begin{aligned}\frac{dy}{dx}&=(x-1)\frac{d\sqrt{x+3}}{dx}+\sqrt{x+3}\,\frac{d(x-1)}{dx}\\&=(x-1)\frac{d(x+3)^{\frac{1}{2}}}{dx}+\sqrt{x+3}\,\frac{d(x-1)}{dx}\\&=(x-1)\frac{1}{2}(x+3)^{-\frac{1}{2}}(1)+\sqrt{x+3}(1)\\&=\frac{1}{2}(x-1)(x+3)^{-\frac{1}{2}}+(x+3)^{\frac{1}{2}}\\&=\frac{1}{2}(x+3)^{-\frac{1}{2}}\left[(x-1)+2(x+3)\right]\\&=\frac{1}{2}(x+3)^{-\frac{1}{2}}(3x+5)\end{aligned}$$



You may wish to view the video titled **Product and Quotient Rule**. This video is program 4 in the *Catch 31*[1] series. Stop the videocassette recorder after you have viewed the first part on the product rule and do the following practice exercise.

[1] *Catch 31* is a title of ACCESS Network.

Now do at least the odd-numbered questions in the following exercise.

Differentiate each of the following.

1. $y=\left(x^2-3x+4\right)\left(2x^2-x+1\right)$
2. $y=\left(x^2+4x-5\right)\left(3x^2+x-1\right)$
3. $y=x^{-2}(x+3)$
4. $y=x^{-5}(x-7)$
5. $y=(x-3)^{-2}(x+1)$
6. $y=(x+4)^{-3}(x-2)$
7. $y=(x-5)\sqrt{x+2}$
8. $y=\sqrt{x-3}\,(x+1)$
9. $y=\sqrt{x-2}\,\sqrt{x+1}$
10. $y=\sqrt{x-3}\,\sqrt{x-5}$

For solutions to **Activity 3**, turn to **Appendix A, Topic 1**.

Activity 4

Use the chain rule and its applications.

Suppose you wish to find $D_x\left(x^2+1\right)^3$.

$$\left(x^2+1\right)^3=\left(x^2+1\right)\left(x^2+1\right)\left(x^2+1\right)$$

Let $u=x^2+1$; then, $D_x u=2x$.

Let $y=u^3$; then, $D_u y=3u^2$.

However, you are to find $D_x y$, not $D_u y$.
There must be some way to bridge the gap between x and y by linking them up through u.

In general if x changes to $x+\Delta x$,
then y changes to $y+\Delta y$.

If $u=f(x)$, then u changes to $u+\Delta u$.

The rate of change in u with respect to x is $\frac{\Delta u}{\Delta x}$.

The rate of change in y with respect to u is $\frac{\Delta y}{\Delta u}$.

Therefore, the rate of change in y with respect to x must be

$\frac{\Delta y}{\Delta x}=\frac{\Delta u}{\Delta x}\bullet\frac{\Delta y}{\Delta u}$.

To find the limit, let $\Delta x \to 0$.

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{\Delta u}{\Delta x}\bullet\frac{\Delta y}{\Delta u}\right)$$

$$=\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\bullet\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u} \qquad \text{(limit theorem)}$$

Since $\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}$, then $\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=\frac{du}{dx}$ and $\Delta u\to 0$ when $\Delta x\to 0$.

Therefore, $\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}=\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}=\frac{dy}{du}$.

Note that $\frac{dy}{dx}=\frac{du}{dx}\bullet\frac{dy}{du}$ is called the **chain rule** for derivatives.

What does the chain rule mean in terms of actually calculating a derivative? Return to the original problem of $y=\left(x^2+1\right)^3$.

If $u=x^2+1$, then $y=u^3$.

If $u=x^2+1$, then $\frac{du}{dx}=2x$.

If $y=u^3$, then $\frac{dy}{du}=3u^2$.

To find $\frac{dy}{dx}$, apply the chain rule.

$$\frac{dy}{dx}=\frac{du}{dx}\bullet\frac{dy}{du}$$

$$\begin{aligned}\frac{dy}{dx}&=(2x)\left(3u^2\right)\\&=(2x)\left[3\left(x^2+1\right)^2\right]\\&=6x\left(x^2+1\right)^2\end{aligned}$$

If you first expand $\left(x^2+1\right)^3$ and take the derivative, you can get the same answer.

The chain rule is very useful. Look at some more examples.

Example 11

Differentiate $y=\left(x^2-2x\right)^5$.

Solution:

Let $u=x^2-2x$; thus, $D_x u=2x-2$.

$$y=u^5$$

$$D_u y=5u^4$$

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\bullet\frac{du}{dx}\\&=5u^4(2x-2)\\&=5\left(x^2-2x\right)^4(2x-2)\end{aligned}$$

The alternative method is used here.

$$y=\left(x^2-2x\right)^5$$

$$\begin{aligned}D_x y&=5\left(x^2-2x\right)^4 D_x\left(x^2-2x\right)\\&=5\left(x^2-2x\right)^4(2x-2)\end{aligned}$$

Example 12

Differentiate $y=\frac{1}{3-x}$.

Solution:

Let $u=3-x$; thus, $D_x u=(-1)$.

$$\begin{aligned}y&=\frac{1}{u}\\&=u^{-1}\end{aligned}$$

$$\frac{dy}{du}=-u^{-2}$$

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\bullet\frac{du}{dx}\\&=-u^{-2}(-1)\\&=u^{-2}\\&=\frac{1}{u^2}\\&=\frac{1}{(3-x)^2}\end{aligned}$$

$$y=(3-x)^{-1}$$

$$\begin{aligned}D_x y&=(-1)(3-x)^{-2}D_x(3-x)\\&=(-1)(3-x)^{-2}(-1)\\&=+(3-x)^{-2}\\&=\frac{1}{(3-x)^2}\end{aligned}$$

Example 13

Differentiate $y = \dfrac{-3}{\sqrt{5x+3}}$.

Solution:

Let $u = (5x+3)$.

$$D_x u = 5$$

$$y = \frac{-3}{u^{\frac{1}{2}}}$$

$$= -3u^{-\frac{1}{2}}$$

$$D_u y = (-3)\left(-\frac{1}{2}\right)u^{-\frac{3}{2}}$$

$$= \frac{3}{2}u^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

$$\therefore D_x y = \frac{3}{2}u^{-\frac{3}{2}} \bullet (5)$$

$$= \frac{15}{2}u^{-\frac{3}{2}}$$

$$= \frac{15}{2}(5x+3)^{-\frac{3}{2}}$$

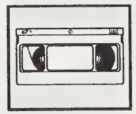
Alternate Method:

$$y = -3(5x+3)^{-\frac{1}{2}}$$

$$D_x y = (-3)\left(-\frac{1}{2}\right)(5x+3)^{-\frac{1}{2}-1} D_x(5x+3)$$

$$= \frac{3}{2}(5x+3)^{-\frac{3}{2}}(5)$$

$$= \frac{15}{2}(5x+3)^{-\frac{3}{2}}$$



For additional instruction on differentiating using the chain rule, you may wish to view the video titled **The Chain Rule**. This video is program 3 in the *Catch 31*[1] series.

[1] *Catch 31* is a title of ACCESS Network.

Now it is time for you to do some practice. Do at least the odd-numbered questions.

Differentiate each of the following.

1. $y=(5x-3)^6$
2. $y=(6x+7)^7$
3. $y=\dfrac{1}{x-5}$
4. $y=\dfrac{4}{x^2+2}$
5. $y=\sqrt{3x-1}$
6. $y=\sqrt{4x+3}$
7. $y=\dfrac{3}{\sqrt{x-2}}$
8. $y=\dfrac{5}{\sqrt{2x+1}}$
9. $y=\left(20-7x^2\right)^5$
10. $y=\dfrac{3}{x^{\frac{5}{3}}}$

For solutions to **Activity 4**, turn to **Appendix A, Topic 1**.

Activity 5

Use the quotient rule and its applications.

The derivative of a quotient can be changed to the derivative of a product. $\left(\text{If } y=\frac{u}{v}, \text{ then it can be changed to } y=uv^{-1}.\right)$ Sometimes it is still convenient to know the rule for the derivative of a quotient.

Here the **quotient rule** will be derived using the product rule.

If $y=\frac{u}{v}$, then $y=uv^{-1}$.

Now use the product rule to find the derivative of y.

$$\begin{aligned}\frac{dy}{dx}&=\left(v^{-1}\right)\frac{du}{dx}+u\frac{d\left(v^{-1}\right)}{dx}\\&=\left(v^{-1}\right)\frac{du}{dx}+(u)(-1)\left(v^{-2}\right)\frac{dv}{dx}\\&=\left(v^{-1}\right)\frac{du}{dx}-uv^{-2}\frac{dv}{dx}\\&=v^{-2}\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\end{aligned}$$

$$\frac{dy}{dx} = \frac{v\,D_x u - u\,D_x v}{v^2}$$

$$D_x\left(\frac{u}{v}\right) = \frac{v\,D_x u - u\,D_x v}{v^2}$$

Note that the derivative of a quotient is **not** equal to the quotient of the derivatives.

This is called the quotient rule and you can apply it as shown in the following examples.

Example 14

Find the derivative for $y = \frac{3x-2}{x^2+1}$.

Solution:

$$\frac{dy}{dx} = \frac{\left(x^2+1\right)D_x(3x-2) - (3x-2)D_x\left(x^2+1\right)}{\left(x^2+1\right)^2}$$

$$= \frac{\left(x^2+1\right)(3) - (3x-2)(2x)}{\left(x^2+1\right)^2}$$

$$= \frac{3x^2+3-6x^2+4x}{\left(x^2+1\right)^2}$$

$$= \frac{-3x^2+4x+3}{\left(x^2+1\right)^2}$$

Example 15

Differentiate $y = \frac{3-x}{\sqrt{x^2-2x}}$.

Solution:

$$\frac{dy}{dx} = \frac{\sqrt{x^2-2x}\,D_x(3-x) - (3-x)D_x\left(\sqrt{x^2-2x}\right)}{x^2-2x}$$

$$= \frac{\left(x^2-2x\right)^{\frac{1}{2}}(-1) - (3-x)\left(\frac{1}{2}\right)\left(x^2-2x\right)^{-\frac{1}{2}}(2x-2)}{\left(x^2-2x\right)}$$

$$= \frac{-\left(x^2-2x\right)^{\frac{1}{2}} - \frac{1}{2}\left(x^2-2x\right)^{-\frac{1}{2}}(3-x)(2x-2)}{\left(x^2-2x\right)}$$

$$= \frac{-\frac{1}{2}\left(x^2-2x\right)^{-\frac{1}{2}}\left[2\left(x^2-2x\right)+(3-x)(2x-2)\right]}{\left(x^2-2x\right)}$$

$$= \frac{\cancel{2x^2} - 4x + 6x - 6 - \cancel{2x^2} + 2x}{-2\left(x^2-2x\right)^{\frac{3}{2}}}$$

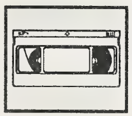
$$= -\frac{(4x-6)}{2\left(x^2-2x\right)^{\frac{3}{2}}}$$

$$= -\frac{(2x-3)}{\left(x^2-2x\right)^{\frac{3}{2}}}$$

Check using the product rule.

$$y=(3-x)\left(x^2-2x\right)^{-\frac{1}{2}}$$

$$\begin{aligned}
D_x y &= (3-x)D_x\left(x^2-2x\right)^{-\frac{1}{2}}+\left(x^2-2x\right)^{-\frac{1}{2}}D_x(3-x)\\
&=(3-x)\left(-\frac{1}{2}\right)\left(x^2-2x\right)^{-\frac{3}{2}}(2x-2)+\left(x^2-2x\right)^{-\frac{1}{2}}(-1)\\
&=-\frac{1}{2}\left(x^2-2x\right)^{-\frac{3}{2}}(2x-2)(3-x)-\left(x^2-2x\right)^{-\frac{1}{2}}\\
&=-\frac{1}{2}\left(x^2-2x\right)^{-\frac{3}{2}}\left[(2x-2)(3-x)+2\left(x^2-2x\right)\right]\\
&=-\frac{1}{2}\left(x^2-2x\right)^{-\frac{3}{2}}\left(6x-\cancel{2x^2}-6+2x+\cancel{2x^2}-4x\right)\\
&=-\frac{(4x-6)}{2\left(x^2-2x\right)^{\frac{3}{2}}}\\
&=\frac{(2x-3)}{\left(x^2-2x\right)^{\frac{3}{2}}}
\end{aligned}$$



You may wish to view the video titled **Product and Quotient Rule**. This video is program 4 in the *Catch 31*[1] series. View the second half of the program on the quotient rule for this activity.

[1] *Catch 31* is a title of ACCESS Network.

Do the following questions. Do at least the odd-numbered questions.

Find $\frac{dy}{dx}$ for each of the following functions using the quotient rule.

1. $y=\dfrac{3x-2}{5x+7}$
2. $y=\dfrac{2-3x+x^2}{5+x+2x^2}$
3. $y=\dfrac{5x-1}{3x-x^2}$
4. $y=\dfrac{x}{\sqrt{2x^2+1}}$
5. $y=\dfrac{2x-3}{\sqrt{x^2-3}}$
6. $y=\dfrac{-5x}{\sqrt{3-x^2}}$

For solutions to **Activity 5**, turn to **Appendix A, Topic 1**.

Activity 6

Determine the derivatives of relations.

Now you will learn about tangents to relations.

You need to make the distinction between a relation and a function. All functions are relations, but not all relations are functions.

Refer to the **vertical-line test** for verification.

The equation $y = f(x) = 2x^2 - 4$ represents a function where y is an **explicit** function of x. The equation $x^2 + y^2 = 9$ illustrates a relation between x and y, where y is an **implicit** function of x. Sometimes such an equation can be solved for y so that y is an explicit function or functions of x. If the equation $x^2 + y^2 = 9$ is solved for y, then $y = \pm\sqrt{9 - x^2}$ where two explicit functions are determined by the implicit equation $x^2 + y^2 = 9$. The two explicit functions of x are $y = \sqrt{9 - x^2}$ and $y = -\sqrt{9 - x^2}$. The graphs of the two explicit functions are the upper and the lower semicircles of the circle $x^2 + y^2 = 9$. The graph of $x^2 + y^2 = 9$ is a circle with a radius of three units and the centre is located at the origin.

It may be difficult to solve an equation for y when the various terms of the equation involve higher exponents. The equation $y^4 + y^3 = x^5 + x$ defines y in terms of x implicitly, but it is not possible to solve for y in terms of x. The method for obtaining the derivative $\left(\frac{dy}{dx}\right)$ of an implicit function without solving the defining equation for the dependent variable (y) in terms of the independent variable (x) is called **implicit differentiation**. The method consists of differentiating both sides of the relation with respect to x and then solving this differentiated relation for $\frac{dy}{dx}$. For problems in this course assume that the given equations define y implicitly as differentiable functions of x. This enables implicit differentiation to be applied to the problems.

Now you can find the derivatives for the following relations using implicit differentiation.

If a vertical line intersects a curve at one distinct point, the curve is described as a **function**.

Example 16

Determine $\frac{dy}{dx}$ for $y = x$.

Solution:

$$y = x$$
$$\frac{dy}{dx} = \frac{dx}{dx}$$
$$\therefore y' = 1 \quad \left(\text{Note that } \frac{dy}{dx} = y'.\right)$$

Example 17

Determine $\frac{dy}{dx}$ for the following.

- $y^2 = x$

  Solution:

$$y^2 = x$$
$$2yy' = 1$$
$$y' = \frac{1}{2y}$$

(Note that y is given implicitly in terms of x.)

- $y = \pm x^{\frac{1}{2}}$

  Solution:

$$y = \pm x^{\frac{1}{2}}$$
$$y' = \pm \frac{1}{2} x^{-\frac{1}{2}}$$
$$= \pm \frac{1}{2x^{\frac{1}{2}}}$$

(Note that y is given explicitly in terms of x.)

By substitution show that $\frac{1}{2y} = \pm \frac{1}{2x^{\frac{1}{2}}}$.

LS	RS
$\frac{1}{2y}$	$\pm \frac{1}{2x^{\frac{1}{2}}}$
$\frac{1}{2\left(\pm x^{\frac{1}{2}}\right)}$	$\pm \frac{1}{2x^{\frac{1}{2}}}$
$\pm \frac{1}{2x^{\frac{1}{2}}}$	$\pm \frac{1}{2x^{\frac{1}{2}}}$

LS = RS

Example 18

Determine $\frac{dy}{dx}$ for $y^3 = x$.

Solution:

$$y^3 = x$$
$$3y^2 y' = 1$$
$$y' = \frac{1}{3y^2}$$

Example 19

Determine $\frac{dy}{dx}$ for $2y^3 + y^2 - 2x^2 = 0$.

Solution:

$$2y^3 + y^2 - 2x^2 = 0$$
$$6y^2 y' + 2yy' - 4x = 0$$
$$6y^2 y' + 2yy' = 4x$$
$$y'(6y^2 + 2y) = 4x$$
$$y' = \frac{4x}{6y^2 + 2y}$$
$$= \frac{2x}{3y^2 + y}$$

Example 20

Determine $\frac{dy}{dx}$ for $y^4 + y^3 = x^5 + x$.

Solution:

In order to determine $\frac{dy}{dx}$ for $y^4 + y^3 = x^5 + x$, both sides of the equation are diffentiated with respect to x.

$$\frac{d}{dx}\left(y^4 + y^3\right) = \frac{d}{dx}\left(x^5 + x\right)$$
$$4y^3 \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 5x^4 + 1$$
$$\left(4y^3 + 3y^2\right)\frac{dy}{dx} = 5x^4 + 1$$
$$\frac{dy}{dx} = \frac{5x^4 + 1}{4y^3 + 3y^2}$$

The graphs of the quadratic relations are good examples where implicit differentiation can be used to obtain $\frac{dy}{dx}$. Consider the following.

- the graph of a circle

$$x^2 + y^2 = 4$$

- the graph of an ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

- the graph of a hyperbola

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

- the graph of a parabola

$$y^2 = 16x$$

It is necessary to point out that certain parabolas and certain hyperbolas are functions. In any case take the derivative of the expression in order to find the slope.

Example 21

Find the derivative $\frac{dy}{dx}$ for the relation

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which represents an ellipse.

Solution:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{2y}{b^2}\frac{dy}{dx} = \frac{-2x}{a^2}$$

$$\frac{dy}{dx} = \frac{-2x}{a^2}\left(\frac{b^2}{2y}\right)$$

$$= \frac{-b^2 x}{a^2 y}$$

This is the value of the derivative. Note that implicit differentiation was used to find $\frac{dy}{dx}$.

If you want to find the slope at (x_1, y_1) for a tangent to an ellipse, you would write the following:

$$\frac{dy}{dx} = \frac{-b^2 x_1}{a^2 y_1}$$

Study the following example.

Implicit differentiation is the process of finding the derivative of each variable with respect to one variable and then solving for the identity of the derivative.

Example 22

- Find the derivative $\frac{dy}{dx}$ of the relation $\frac{x^2}{5} - \frac{y^2}{1} = 1$.

  Solution:

$$\frac{x^2}{5} - \frac{y^2}{1} = 1$$

$$\frac{2x}{5} - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2x}{5}\left(\frac{1}{2y}\right)$$

$$= \frac{x}{5y}$$

  The derivative of the relation is $\frac{x}{5y}$.

- What would be the slope of its tangent if the curve passes through $(5,2)$?

  Solution:

  The slope of a tangent line at point $(5,2)$ is as follows:

$$\frac{dy}{dx} = \frac{x}{5y}$$

$$= \frac{5}{5(2)}$$

$$= \frac{1}{2}$$

  The slope of the tangent would be $\frac{1}{2}$.

Try another example.

Example 23

- Find the derivative of y with respect to x for the relation $2x^2 + xy - y^2 = -7$.

  Solution:

$$2x^2 + xy - y^2 = -7$$

$$4x + y(1) + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} - 2y\frac{dy}{dx} = -4x - y$$

$$(x - 2y)\frac{dy}{dx} = -(4x + y)$$

$$\frac{dy}{dx} = \frac{-(4x + y)}{x - 2y}$$

  The derivative of the relation is $\frac{-(4x+y)}{x-2y}$.

- Find the value of the derivative $\frac{dy}{dx}$ at $B(-2,3)$ on the curve, and state where $\frac{dy}{dx}$ does not exist.

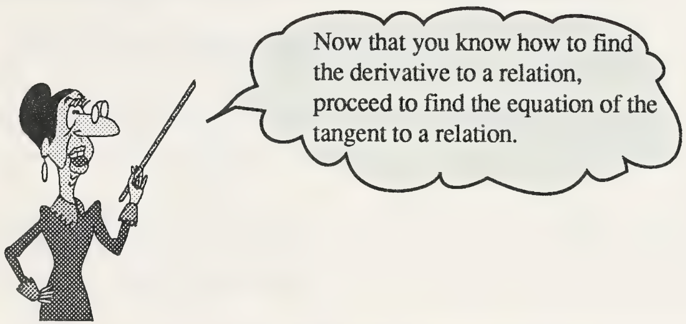
Solution:

$$\frac{dy}{dx}=\frac{-(4x+y)}{x-2y}$$

The derivative at $B(-2,3)$ is as follows:

$$\begin{aligned}\frac{dy}{dx}&=\frac{-[4(-2)+3]}{(-2)-2(3)}\\&=\frac{-8+3}{2+6}\\&=\frac{-5}{8}\end{aligned}$$

The value of the derivative at B is $\frac{-5}{8}$. The derivative is not defined at $x-2y=0$. That is when $x=2y$.



Example 24

Find the equation of the tangent line to the relation $\frac{x^2}{4}+\frac{y^2}{9}=2$.

The line passes through the point $P(2,-3)$.

Solution:

$$\begin{aligned}\frac{x^2}{4}+\frac{y^2}{9}&=2\\\frac{2x}{4}+\frac{2y}{9}\frac{dy}{dx}&=0\\\frac{2y}{9}\frac{dy}{dx}&=-\frac{x}{2}\\\frac{dy}{dx}&=\frac{-x}{2}\left(\frac{9}{2y}\right)\\&=\frac{-9x}{4y}\end{aligned}$$

The slope of the line at $P(2,-3)$ is as follows:

$$\begin{aligned}\frac{dy}{dx}&=\frac{-9(2)}{4(-3)}\\&=\frac{-18}{-12}\\&=\frac{18}{12}\\&=\frac{3}{2}\end{aligned}$$

In order to find the equation of the tangent line, substitute $m=\frac{3}{2}$ and $P(2,-3)$ in the slope-point formula.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-(-3) &= \frac{3}{2}(x-2) \\ y &= \frac{3}{2}x-6 \\ 3x-2y-12 &= 0 \end{aligned}$$

Therefore, the equation of the tangent line is $3x-2y-12=0$.

The questions involving relations and tangents are not always this easy. You should try one that is in the form of a quotient.

Example 25

Find the of the tangent to the graph of $y=\frac{x+3}{x+2}$ at the point $Q(-1,2)$. Find the equation for another tangent if it is parallel to the first tangent.

Solution:

Note that $y=\frac{x+3}{x+2}$ becomes $xy + 2y = x + 3$.

$$\begin{aligned} y(1)+x\frac{dy}{dx}+2\frac{dy}{dx} &= 1 \\ (x+2)\frac{dy}{dx} &= 1-y \\ \frac{dy}{dx} &= \frac{1-y}{x+2} \end{aligned}$$

The slope at $Q(-1,2)$ is as follows:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1-y}{x+2} \\ &= \frac{1-2}{-1+2} \\ &= -1 \end{aligned}$$

Use the slope-point formula to find the equation of the tangent line.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-2 &= -1[x-(-1)] \\ y-2 &= -x-1 \\ x+y-1 &= 0 \end{aligned}$$

The equation of the tangent at $Q(-1,2)$ is $x+y-1=0$.

For $\frac{dy}{dx}=-1$ other values of x and y may exist. If there is a tangent parallel to the first line, then its slope is -1.

Therefore, $-1 = \frac{1-y}{x+2}$ where $y = \frac{x+3}{x+2}$.

$$-1 = \frac{1-\left(\frac{x+3}{x+2}\right)}{x+2}$$

$$-1 = \frac{x+2-(x+3)}{(x+2)^2}$$

$$\frac{x+2-x-3}{(x+2)^2} = -1$$

$$\frac{-1}{(x+2)^2} = -1$$

$$(x+2)^2 = 1$$

$$x+2 = \pm 1$$

$$\therefore x = -3 \text{ or } x = -1$$

For $x=-1$, $y=2$. Therefore, the first point of tangency given is $Q(-1,2)$.

For $x=-3$, $y = \frac{-3+3}{-3+2} = \frac{0}{-1} = 0$.

The second point of tangency is $(-3, 0)$. The equation of the tangent at $(-3,0)$ is $y = -x-3$.

You probably know that circles are good examples of relations. You also know that if you are given a point outside of a circle, you can draw two tangent lines to the figure.

In the next example you will find two tangents to a circle at given points.

Example 26

Find the tangents to the relation $x^2 + y^2 = 169$ (a circle) at the points $P(5,12)$ and $Q(5,-12)$. At what point T do they intersect?

Solution:

$$x^2 + y^2 = 169$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

At $P(5,12)$ the slope $\frac{dy}{dx} = -\frac{5}{12}$.

Thus, substitute $P(5,12)$ and $m = \frac{-5}{12}$ in the slope-point formula.

$$y-12 = \frac{-5}{12}(x-5)$$

$$5x+12y = 169$$

The tangent equation at P is $5x + 12y = 169$. 1

At $Q(5,-12)$ the slope $\frac{dy}{dx} = -\frac{5}{-12} = \frac{5}{12}$..

Therefore, substitute $Q(5,-12)$ and $m = \frac{5}{12}$ in the point-slope formula.

$$y-(-12) = \frac{5}{12}(x-5)$$
$$5x - 12y = 169$$

The tangent equation at Q is $5x - 12y = 169$. (2)

In order to find point T where the tangents meet, solve equations (1) and (2).

$$\begin{array}{rrl} & 5x+12y=169 & (1) \\ & 5x-12y=169 & (2) \\ \hline (1)+(2): & 10x=338 & \\ & x=33.8 & \end{array}$$

When $x = 33.8$, $y = 0$.

Therefore, the point at where the tangents meet is $T(33.8, 0)$.

Now that you have examined tangents as they relate to relations and functions, try the following questions.

1. Find y' for the following relations.

   a. $3xy + 2x^2 + y^2 = 2$

   b. $y^3 - 3xy^2 - 2x^2y + x = 2$

   c. $\frac{x^{\frac{1}{3}}}{a} - \frac{y^{\frac{1}{3}}}{a} = 1$

   d. $xy^2 = 2x - \frac{1}{x}$

2. Write the equation of the tangent to the graph of relation $x^2 + 4xy - 2y = 3$ at the point $P(1,1)$.

For solutions to **Activity 6**, turn to **Appendix A, Topic 1**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

Extra Help

The differentiation rules are very useful. It is important to remember them. (Recall that u and v are functions of x.)

- sum or difference rule: $D_x(u \pm v) = D_x u \pm D_x v$

  For example, differentiate $y = \left(x^{-2} - 3x^5\right)$.

$$\frac{dy}{dx} = D_x\left(x^{-2}\right) - D_x\left(3x^5\right)$$
$$= -2x^{-3} - 15x^4$$

- power rule: $D_x x^n = nx^{n-1}$

  For example, differentiate $y = x^{-\frac{1}{2}}$.

$$\frac{dy}{dx} = \left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$$
$$= -\frac{1}{2}x^{-\frac{3}{2}}$$

- product rule: $D_x(vu) = v\,D_x u + u\,D_x v$

  For example, differentiate $\left(x^3 - 1\right)(x+5)$.

$$D_x\left(x^3-1\right)(x+5) = \left(x^3-1\right)D_x(x+5) + (x+5)D_x\left(x^3-1\right)$$
$$= \left(x^3-1\right)(1) + (x+5)\left(3x^2\right)$$
$$= \left(x^3-1\right) + 3x^2(x+5)$$
$$= 4x^3 + 15x^2 - 1$$

- chain rule: $\frac{dy}{dx}=\frac{dy}{du}\bullet\frac{du}{dx}$

For example, differentiate $y=\left(x^3-2x+1\right)^5$.

Let $u=x^3-2x+1$.

$$D_x u=3x^2-2$$

$$y=u^5$$

$$\frac{dy}{du}=5u^4$$

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\bullet\frac{du}{dx}\\&=\left(5u^4\right)\left(3x^2-2\right)\\&=5\left(x^3-2x+1\right)^4\left(3x^2-2\right)\end{aligned}$$

- quotient rule: $D_x\left(\frac{u}{v}\right)=\frac{v\,D_x u-u\,D_x v}{v^2}$

For example, differentiate $\frac{x-5}{x^2+7}$.

$$\begin{aligned}D_x\left(\frac{x-5}{x^2+7}\right)&=\frac{\left(x^2+7\right)D_x(x-5)-(x-5)D_x\left(x^2+7\right)}{\left(x^2+7\right)^2}\\&=\frac{\left(x^2+7\right)(1)-(x-5)(2x)}{\left(x^2+7\right)^2}\\&=\frac{x^2+7-2x^2+10x}{\left(x^2+7\right)^2}\\&=\frac{-x^2+10x+7}{\left(x^2+7\right)^2}\end{aligned}$$

Now try the following questions.

Differentiate each of the following.

1. $y = x^{10}$
2. $y = x^{-3}$
3. $y = 3x^{5}$
4. $y = 2x^{-2}$
5. $y = 3x^{2} - 4x + 7$
6. $y = (x-3)(x^{2}+5)$
7. $y = (7-2x)^{6}$
8. $y = \dfrac{2x-3}{3x+8}$

For solutions to **Extra Help**, turn to **Appendix A, Topic 1**.

Extensions

Can you prove the power rule?

If $y = x^{n}$, then $D_x y = nx^{n-1}$, where $x \in R$ and $n \in Q$. (Note that R represents the real numbers and Q represents the rational numbers.)

To prove this theorem you have to consider two cases.

Case 1: *n* is positive

Let $n = \frac{a}{b}$, where $a, b \in N$.

$$y = x^{\frac{a}{b}}$$

Raise both sides to the power of b.

$$y^{b} = x^{a} \qquad (1)$$

If that x changes by a small quantity Δx, then y also changes by Δy.

$$(y+\Delta y)^{b} = (x+\Delta x)^{a} \qquad (2)$$

$$(2)-(1):\ (y+\Delta y)^{b} - y^{b} = (x+\Delta x)^{a} - x^{a}$$

Factor both sides. The factors of $(y+\Delta y)^b - y^b$ are found in the following way:

$$\begin{array}{r|l}
 & \overbrace{(y+\Delta y)^{b-1} + (y+\Delta y)^{b-2}y + \cdots + y^{b-1}}^{b \text{ terms}} \\ \hline
y+\Delta y - y & (y+\Delta y)^b + 0 + 0 \cdots - y^b \\
 & \underline{(y+\Delta y)^b - (y+\Delta y)^{b-1}y} \\
 & \qquad (y+\Delta y)^{b-1}y \\
 & \qquad \underline{(y+\Delta y)^{b-1}y - (y+\Delta y)^{b-2}y^2} \\
 & \qquad\qquad \cdots \\
 & \qquad\qquad \cdots \\
 & \qquad\qquad (y+\Delta y)y^{b-1} - y^b \\
 & \qquad\qquad \underline{(y+\Delta y)y^{b-1} - y^b} \\
 & \qquad\qquad\qquad 0
\end{array}$$

Therefore, $(y+\Delta y)^b - y^b = (y+\Delta y - y)\left[(y+\Delta y)^{b-1} + \cdots + y^{b-1}\right]$.

The factors of $(x+\Delta x)^a - x^a$ are found in the same way such that

$(x+\Delta x)^a - x^a = (x+\Delta x - x)\left[(x+\Delta x)^{a-1} + \cdots + x^{a-1}\right]$.

Now substitute these factors into the previous equation.

$$(y+\Delta y - y)\left[(y+\Delta y)^{b-1} + \cdots + y^{b-1}\right] = (x+\Delta x - x)\left[(x+\Delta x)^{a-1} + \cdots + x^{a-1}\right]$$

$$\Delta y\left[(y+\Delta y)^{b-1} + \cdots + y^{b-1}\right] = \Delta x\left[(x+\Delta x)^{a-1} + \cdots + x^{a-1}\right]$$

$$\frac{\Delta y}{\Delta x} = \frac{\left[(x+\Delta x)^{a-1} + \cdots + x^{a-1}\right]}{\left[(y+\Delta y)^{b-1} + \cdots + y^{b-1}\right]}$$

Take the limit as $\Delta x \to 0$.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\overbrace{\left[(x+\Delta x)^{a-1} + \cdots + x^{a-1}\right]}^{a \text{ terms}}}{\underbrace{\left[(y+\Delta y)^{b-1} + \cdots + y^{b-1}\right]}_{b \text{ terms}}}$$

$$\frac{dy}{dx} = \frac{ax^{a-1}}{by^{b-1}}$$

$$= \frac{a}{b} \bullet \frac{x^{a-1}}{y^{b-1}}$$

Since $y = x^{\frac{a}{b}}, y^{b-1} = \left(x^{\frac{a}{b}}\right)^{b-1} = x^{\frac{ab-a}{b}}$.

$$\frac{dy}{dx} = \frac{a}{b} \bullet \frac{x^{a-1}}{x^{\frac{ab-a}{b}}}$$

$$= \frac{a}{b} \bullet x^{(a-1)-\frac{ab-a}{b}}$$

$$= \frac{a}{b} x^{\frac{a}{b}-1}$$

Since $n = \frac{a}{b}$, $\frac{dy}{dx} = nx^{n-1}$.

Case 2: *n* is negative

Let $n = -\frac{a}{b}$, where $a, b \in I$ and $a, b > 0$.

$$y = x^{-\frac{a}{b}}$$

Raise both sides to the power of b.

$$y^b = x^{-a} = \frac{1}{x^a} \qquad (1)$$

If x changes by a small quantity Δx, then y changes by Δy.

$$(y+\Delta y)^b = \frac{1}{(x+\Delta x)^a} \qquad (2)$$

$$(2)-(1):\ (y+\Delta y)^b - y^b = \frac{1}{(x+\Delta x)^a} - \frac{1}{x^a}$$

$$= \frac{x^a - (x+\Delta x)^a}{x^a (x+\Delta x)^a}$$

$$= -\frac{(x+\Delta x)^a - x^a}{x^a (x+\Delta x)^a}$$

$$(a-1) - \frac{ab-a}{b} = \frac{b(a-1) - ab + a}{b}$$

$$= \frac{\cancel{ab} - b - \cancel{ab} + a}{b}$$

$$= \frac{a}{b} - \frac{b}{b}$$

$$= \frac{a}{b} - 1$$

Factor both sides.

$$(y+\Delta y-y)\left[(y+\Delta y)^{b-1}+\cdot\cdot\cdot+y^{b-1}\right]=-\frac{(x+\Delta x-x)\left[(x+\Delta x)^{a-1}+\cdot\cdot\cdot+x^{a-1}\right]}{x^{a}(x+\Delta x)^{a}}$$

$$\Delta y\left[(y+\Delta y)^{b-1}+\cdot\cdot\cdot+y^{b-1}\right]=-\frac{\Delta x\left[(x+\Delta x)^{a-1}+\cdot\cdot\cdot+x^{a-1}\right]}{x^{a}(x+\Delta x)^{a}}$$

$$\frac{\Delta y}{\Delta x}=-\frac{\left[(x+\Delta x)^{a-1}+\cdot\cdot\cdot+x^{a-1}\right]}{\left[(y+\Delta y)^{b-1}+\cdot\cdot\cdot+y^{b-1}\right]\left[x^{a}(x+\Delta x)^{a}\right]}$$

Take the limit as $\Delta x \to 0$.

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{-\left[\overbrace{(x+\Delta x)^{a-1}+\cdot\cdot\cdot+x^{a-1}}^{a\text{ terms}}\right]}{\left[(y+\Delta y)^{b-1}+\cdot\cdot\cdot+y^{b-1}\right]\left[x^{a}(x+\Delta x)^{a}\right]}$$

$$=\frac{-ax^{a-1}}{\left(by^{b-1}\right)\left(x^{2a}\right)}$$

$$=\frac{-ax^{a-1}}{b\left(x^{\frac{-a}{b}}\right)^{b-1}x^{2a}}$$

$$=\frac{-ax^{a-1}}{bx^{-a+\frac{a}{b}}\left(x^{2a}\right)}$$

$$=-\frac{a}{b}x^{a-1-\frac{ab+a}{b}}$$

$$=-\frac{a}{b}x^{-\frac{a}{b}-1}$$

$$-a+\frac{a}{b}+2a=\frac{-ab+a+2ab}{b}$$

$$=\frac{ab+a}{b}$$

$$a-1-\frac{ab+a}{b}=\frac{\cancel{ab}-b-\cancel{ab}-a}{b}$$

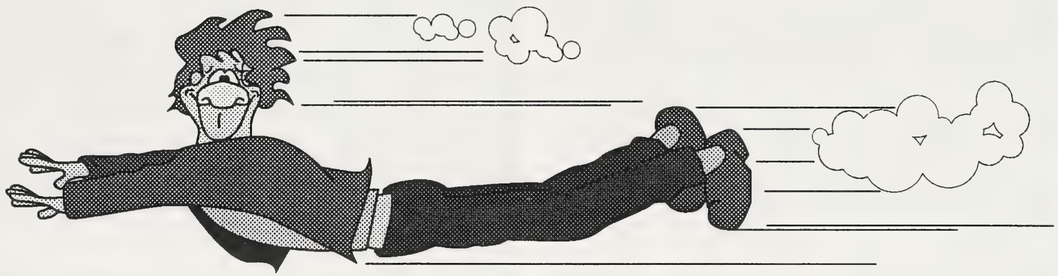
$$=-\frac{a}{b}-1$$

Since $n = -\frac{a}{b}$, $\frac{dy}{dx} = nx^{n-1}$.

Now try a similar problem.

Use the same method to prove that $\frac{dy}{dx} = nx^{n-1}$ if $y = x^n$, where $x \in R$ and $n \in N$.

For solutions to **Extensions**, turn to **Appendix A, Topic 1**.



Topic 2 Graphing Polynomial Functions

Introduction

When sketching the graph of a polynomial function, use all the information available. By using the first and second derivatives of a function, you can find the maximum points, the minimum points, and the points of inflection. These will help determine the position, direction, and concavity of the graph of a polynomial function. In this topic you are going to learn about these concepts.

What Lies Ahead

Throughout the topic you will learn to

1. apply the first derivative to determine coordinates of the turning points, and define and determine the critical values
2. define and determine the second derivative of a function, the points of inflection, and the characteristics of the turning points
3. determine the concavity, the position, and the direction of the graph of a polynomial function

Now that you know what to expect, turn the page to begin your study of graphing polynomial functions.

Exploring Topic 2

Activity 1

Apply the first derivative to determine coordinates of the turning points, and define and determine the critical values.

After studying polynomial functions, you learned to sketch graphs of functions using the intercepts and the sign of the coefficient of the variable of the highest order. To refresh your memory, look at the following example.

Example 1

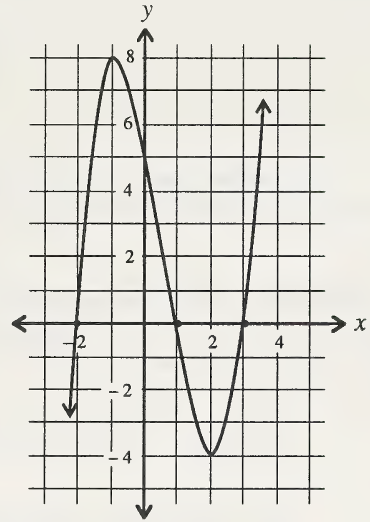
Sketch the graph of $y = x^3 - 2x^2 - 5x + 6$.

Solution:

Using your skills with the factor theorem, you obtain $y = (x-1)(x+2)(x-3)$.

When $x = 0$, $y = 6$ (y-intercept).
When $y = 0$, $x = 1, -2$, or 3 (x-intercepts).

You also know that the graph of a cubic curve with a positive leading coefficient generally goes upward from the third quadrant and has two turning points as it moves to the right. For example, the graph goes up, then down, and then up again.



Look at this relationship between the leading coefficient and direction of the curve defined by a polynomial function more closely.

Recall: The factor theorem states that a polynomial $P(x)$ has the binomial $(x-a)$ as a factor only if $P(a) = 0$.

What happens to the y-values as $x \to \pm\infty$? This is the essence of the problem.

Consider the function $y = x^3 - 2x^2 - 5x + 6$.

In order to get an idea of the behaviour of this function for larger values of x, you will evaluate the function for $x = 10^{100}$ or a googol.

$$\begin{aligned} y &= \left(10^{100}\right)^3 - 2\left(10^{100}\right)^2 - 5\left(10^{100}\right) + 6 \\ &= 10^{300} - 2\left(10^{200}\right) - 5\left(10^{100}\right) + 6 \end{aligned}$$

10^{300} is a large positive number (1 followed by 300 zeros).

How significant are the other terms in this expression relative to 10^{300}?

Similarly, as $x \to +\infty$, the terms other than x^3 in this expression become less and less significant in evaluating the function.

Thus, as $x \to +\infty$ the leading term of the polynomial function takes over and the behaviour of the curve is dependent on that leading term. Similarly, as $x \to -\infty$ the leading term is the controlling term of the function.

Consider the function $y = x^3 - 2x^2 - 5x + 6$.

As $x \to +\infty$, $x^3 \to +\infty$; thus, $y \to +\infty$.
The curve goes upward to the right.

As $x \to -\infty$, $x^3 \to -\infty$, so $y \to -\infty$.
The curve goes downward to the left.

Another point of interest is the position of the curve relative to the x-axis. For what values of x is the graph above the axis? Where is it below?

The answers to these questions are important to the solution of polynomial inequalities.

Note that $x \to \infty$ is read **as** x **approaches infinity**.

1 googol $= 10^{100}$

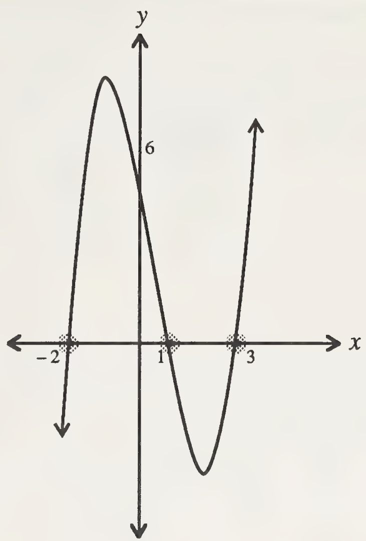
1 googolplex $= 10^{\text{googol}}$

The controlling term is the term of the highest order.

When the polynomial function $y = x^3 - 2x^2 - 5x + 6$ is evaluated for a real value of x, there are three possibilties.

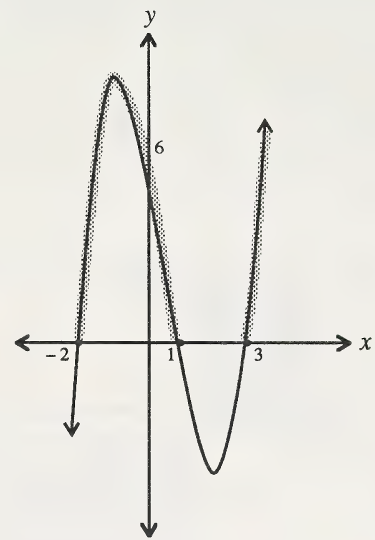
- If $y = 0$, then the x-intercepts of the graph are – 2, 1, and 3.

$$y = 0$$
$$\therefore x^3 - 2x^2 - 5x + 6 = 0$$



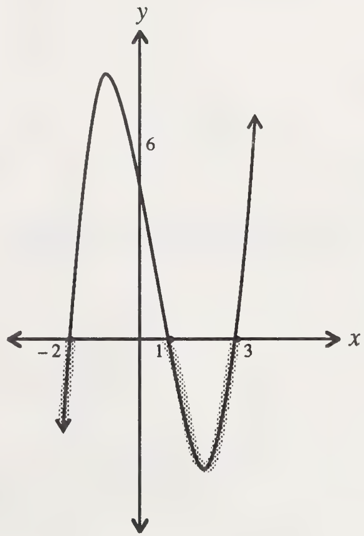
- If y is positive $(y > 0)$, then the graph is above the x-axis.

$$y > 0$$
$$\therefore x^3 - 2x^2 - 5x + 6 > 0$$



- If y is negative $(y < 0)$, then the graph is below the x-axis.

$$y < 0$$
$$\therefore x^3 - 2x - 5x + 6 < 0$$



When deciding whether intervals for the function are positive or negative, it is important that you know where the function is equal to zero (x-intercepts). For the purpose of this unit you shall call these points **critical points** or **critical values**.

In order to sketch the graph of a polynomial function, you need to do the following:

- Determine the y-intercept.
- Determine the critical values (the x-intercepts).
- Analyse the function to determine where the graph is above or below the x-axis.
- Identify the behaviour of the graph as $x \to \pm\infty$.
- Use this information to sketch the graph.

The following example shows the five step procedure.

Example 2

Sketch the graph of $y = (x+2)^2(x-1)$.

Solution:

To find the y-intercept, let $x = 0$.

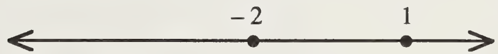
$$\begin{aligned} y &= (0+2)^2(0-1) \\ &= -4 \end{aligned}$$

The y-intercept is -4.

To find the critical values of the function, let $y=0$.

$$0=(x+2)^2(x-1)$$
$$x=-2 \text{ or } x=1$$

Set up a number line and divide it into intervals using the critical values. (Two critical points define three intervals).



Choose test values in the intervals defined, and determine the sign of the function. For example, in the interval $x<-2$, you might choose $x=-3$. Substituting, you get the following:

$$y=(-3+2)^2(-3-1)$$
$$=(-1)^2(-4)$$
$$=-4$$

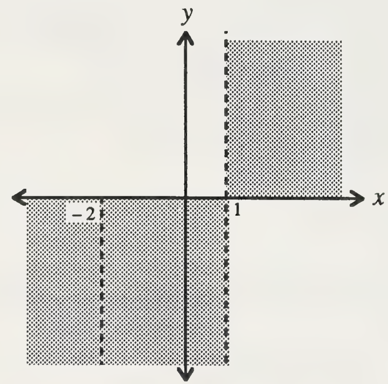
The point of interest here is that the result is negative; thus, $y<0$. This is the case for all $x<-2$ since the graph is continuous over this interval. Since there are no x-intercepts in that interval, the graph cannot get to the other side of the axis; therefore, if the function is negative at one point in the interval, it must be negative for all points in that interval. If you analyse the other intervals, you put the sign of the results on the number line as shown on the following page.

This tells you that the graph is below the x-axis for $x<-2$ and $-2<x<1$, and is above the x-axis for $x>1$. This identifies regions where the curve is located.

$$y=(x+2)^2(x-1)$$

The critical values are -2 and 1.

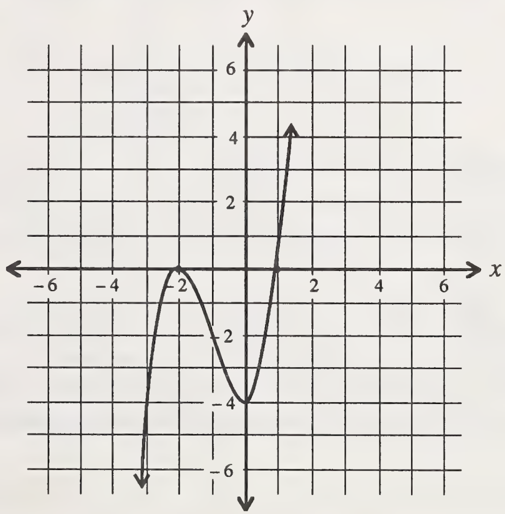
Interval	Test Point	$(x+2)^2(x-1)=y$
$x<-2$	-3	$(-)^2(-)=-$
$-2<x<1$	0	$(+)^2(-)=-$
$x>1$	2	$(+)^2(+)=+$



Points on the curve cannot be outside the shaded region.

As $x \to \pm\infty$ the highest degree term in the expanded form determines the behaviour of the graph. In this case the leading term would be $+x^3$. Thus, as $x \to +\infty$, $y \to +\infty$; and as $x \to -\infty$, $y \to -\infty$.

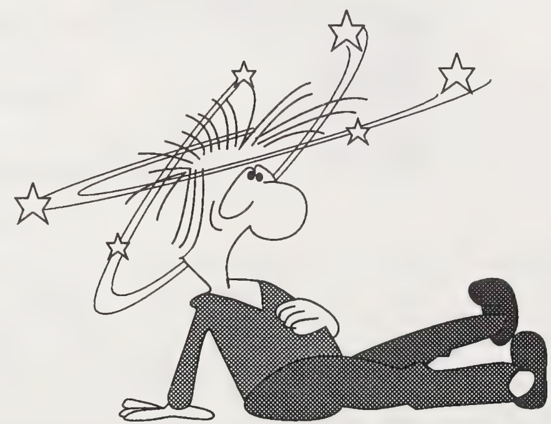
Using all of this information, sketch the graph. Notice how the information you have forces you to draw a curve that touches the x-axis at -2 and intersects the x-axis at 1. This also confirms what you know about multiplicities or more than one of the same factor. If there is an even multiplicity $\left[(x+2)^2\right]$, the curve will only touch the x-axis. If there is an odd multiplicity $\left[(x+2)^3\right]$, the curve will cut the x-axis.



If you have successfully completed polynomial functions, you probably know most of the information in the discussion to this point. However, there might be a point or two that you didn't know. You should make a note of those points.

In Unit 1 there was much discussion of the first derivative and emphasis that it was an expression that determined the slope of the curve defined by the function. The local extremes of a function are determined by finding where the slope is zero. This can be useful to you in enhancing your picture of the function.

Take a look at another polynomial function and its graph.



Example 3

Sketch the graph of $y = x^3 + 3x^2 - 9x - 27$.

Solution:

To find the y-intercept, let $x = 0$.

$$y = (0)^3 + 3(0)^2 - 9(0) - 27$$
$$y = -27$$

The y-intercept is $(0, 27)$.

To find the critical values, let $y = 0$.

$$0 = x^3 + 3x^2 - 9x - 27$$
$$0 = (x+3)^2(x-3)$$
$$x = -3 \text{ or } x = 3$$

The critical values are $(-3, 0)$ and $(3, 0)$.

The behaviour of the graph is as follows:

As $x \to +\infty, y \to +\infty$.
As $x \to -\infty, y \to -\infty$.

Investigate the first derivative and its implications for the graph.

$$y = x^3 + 3x^2 - 9x - 27$$
$$y' = 3x^2 + 6x - 9$$

The zeros of the first derivative are critical. Therefore, find the critical values of the first derivative; then, analyse the first derivative in much the same way as shown for the function.

To find the critical values of the first, let $y' = 0$.

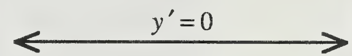
$$0 = 3x^2 + 6x - 9$$
$$0 = 3(x+3)(x-1)$$

The critical values of the first derivative are $x = -3$ and $x = 1$. By substitution into the original function you find the two points $(-3, 0)$ and $(1, -32)$. The two points are the turning points of the graph.

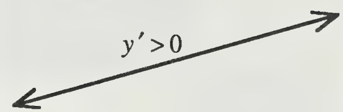
Note that y' is the first derivative.

When the first derivative is evaluated for a given value of x, the only possibilities are as follows:

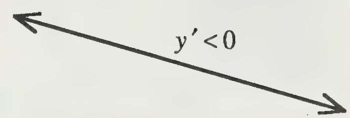
- If $y' = 0$, the tangent is horizontal.



- If $y' > 0$, the tangent rises to the right.

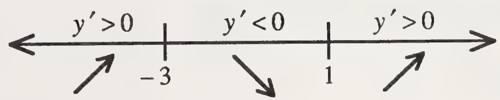


- If $y' < 0$, the tangent falls to the right.



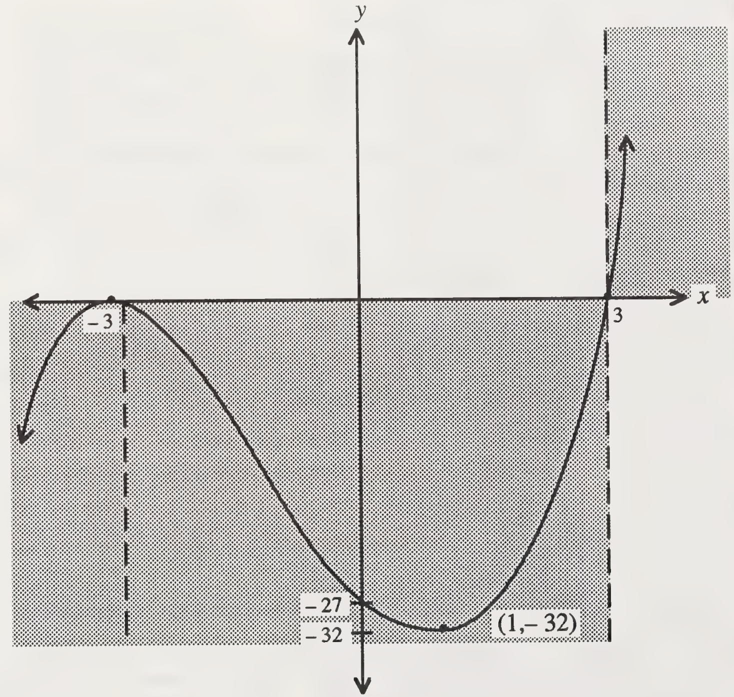
Analyse the first derivative.

Interval	Test Point	$3(x+3)(x-1)=y'$
$x<-3$	-10	$(+)(-)(-)=+$
$-3<x<1$	0	$(+)(+)(-)=-$
$x>1$	2	$(+)(+)(+)=+$



In the interval $x<-3$ the slope is positive. The slope is also positive when $x>1$. This means that as x increases, the function increases. The function is increasing in the intervals $x<-3$ and $x>1$ since $y'>0$ in these intervals. The function is decreasing in the interval $-3<x<1$ since $y'<0$ in this interval.

How do you know which turning point is a maximum or minimum value? If you study the analysis of the derivative, you see at point $(-3, 0)$ that 0 is a local maximum since the slope before -3 is positive and the slope after -3 is negative. This is the condition that yields a local maximum. Note the conditions around $x=1$. These conditions are required for a local minimum. The local minimum value is $y=-32$.



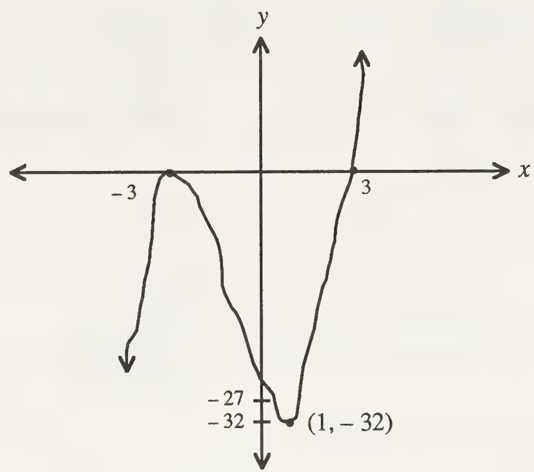
By the term **conditions around $x = 1$** you are referring to the neighbourhood of the point on the curve where $x = 1$.

If the first derivative goes from positive to negative in the neighbourhood from left to right, you have a local maximum.

If the first derivative goes from negative to positive in the neighbourhood from left to right, you have a local minimum.

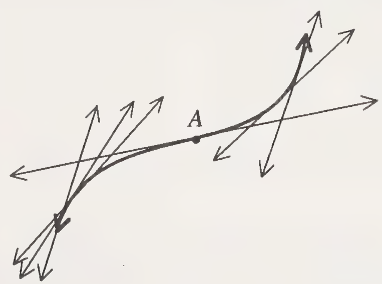
There is at least one more question to be asked about the graphs of polynomial functions. Consider the graph produced in Example 3.

From our analysis of the function and its derivative you know the axis intercepts, the regions on the coordinate plane where you can find the graph, and the intervals where the function is increasing and decreasing.



This graph conforms to the information as well as the graph you sketched. Which is correct and why?

You say the smooth curve is correct because that is what you were told when you studied polynomial functions. Look more closely at the following curve. What happens to the slope at certain points on the curve?



The curve is getting shallower up to point A. Notice the slope is decreasing. The curve is getting steeper after point A. Notice the slope is increasing. Changes in the curvature of the curve are caused by changes in the way the slope behaves.

What you need is some way of determining whether the slope is increasing or decreasing over a certain interval.

Think about that for a minute. Does it sound familiar? Have you ever done anything like that before?

Analysis of the first derivative of the function tells you where the function is increasing or decreasing.

If the first derivative tells you how the function behaves, then the derivative of the first derivative (known as the **second derivative**) should tell you how the slope behaves.

You will get a chance to look more carefully at the second derivative in the next section.

Now try a few questions that deal with the first derivative.

1. Do the following questions for a and b.

   a. $f(x) = -x^3 + 3x$

      $f'(x) = -3x^2 + 3$

   b. $g(x) = (x-2)^2(x+1)$

      $g'(x) = 3x(x-2)$

   i. Verify that the given derivative is the derivative of the function.

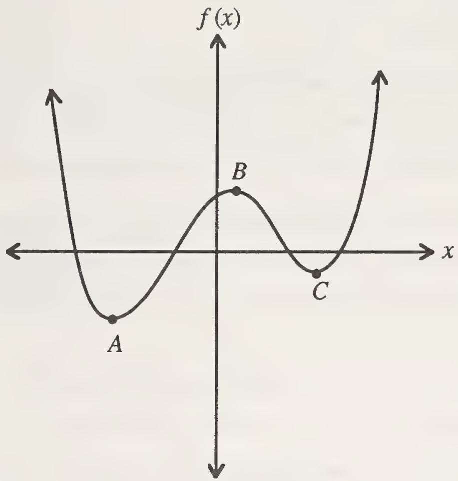
   ii. Determine the critical values of the function and its derivative.

   iii. Analyze the function and its derivative.

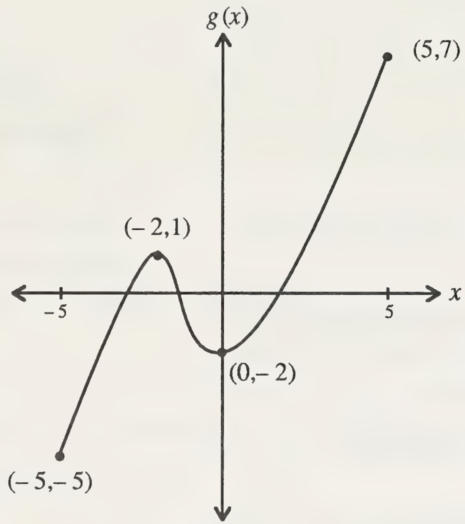
   iv. Use your analysis to draw the graph of the functions in a and b. Graph paper is provided in **Appendix B**.

2. Determine where $y = (x-1)^3(x+2)^2$ is increasing and where it is decreasing.

3. With reference to the following curve, explain the difference between local and absolute minimum values. Does the function $y = f(x)$ have a maximum value?



4. The following graph shows a function $g(x)$ which is defined over a restricted domain $-5 \le x \le 5$. Identify and distinguish between local and absolute extremes.



For solutions to **Activity 1**, turn to **Appendix A, Topic 2**.

A restricted domain includes end points. $x < 2$ is not restricted, but $-1 \le x \le 1$ is restricted.

Activity 2

Define and determine the second derivative of a function, the points of inflection, and the characteristics of the turning points.

Common notations for the **second derivative** are as follows:

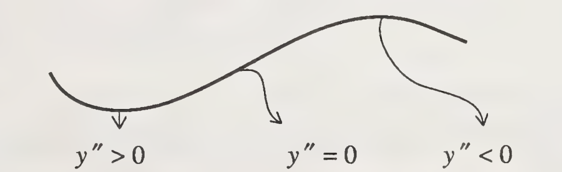
$$\frac{d^2y}{dx^2} \qquad y'' \qquad f''(x) \qquad D_x^2 y$$

The second derivative is the derivative of the first derivative.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d}{dx}(y)\right]$$

The first derivative y' represents the rate of change of $y = f(x)$ with respect to x. The second derivative y'' gives the rate of change of the first derivative with respect to x.

The second derivative is also a test of concavity. If y'' is positive, the graph is **concave up**. If y'' is negative, the graph is **concave down**. If $y'' = 0$, the graph often has a **point of inflection**.



It is important to know where the second derivative is positive and where it is negative. That means a **critical value** of the second derivative is where the second derivative is zero.

What is the significance of this point? If the slope prior to this was increasing and after that point it is decreasing or vice-versa, you have found a point which is called a point of inflection.
An inflection point can only occur where there is a change in the concavity of the graph of the function.

A point of inflection occurs at any point on the curve defined by the function $y = f(x)$ that satisfies the following conditions:

- $f''(x) = 0$ (The second derivative of the function is zero at that point.)
- The first derivative $f'(x)$, or slope, remains positive or remains negative through that point. There is no change of direction at that point.
- $f''(x)$ changes sign as the function passes through the point.

Examine the following two diagrams showing a local maximum point and a local minimum point respectively.

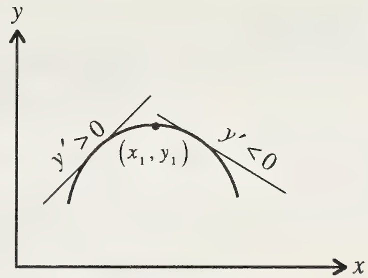


Diagram A

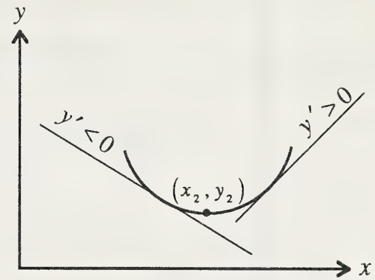
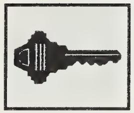


Diagram B



The conditions at a **local maximum** are as follows:

- $y' = 0$
- $y'' < 0$

In Diagram A a local maximum point is represented by point (x_1, y_1) on the curve $y = f(x)$. The slope y' of the curve is positive when $x < x_1$ and negative when $x > x_1$. At $x = x_1$ the slope $y' = 0$. The value of y' decrease as x approaches x_1; thus, the rate of change of y' is negative as x increases. Therefore, y'' is negative at $x = x_1$.

The conditions at a **local minimum** are as follows:

- $y' = 0$
- $y'' > 0$

In Diagram B a local minimum point is represented by point (x_2, y_2) on the curve $y = f(x)$. The slope y' of the curve is negative when $x < x_2$ and positive when $x > x_2$. At $x = x_2$ the slope $y' = 0$. The value of y' increases as x approaches x_2; thus, the rate of change of y' is postive as x increases. Therefore, y'' is positive at $x = x_2$.

Example 4

Determine the point(s) of inflection of the graph $y = x^3 - 3x^2 + 4$.

Solution:

$y' = 3x^2 - 6x$
Let $y' = 0$.

$3x^2 - 6x = 0$
$3x(x-2) = 0$
$x = 0 \text{ or } x = 2$

$y'' = 6x - 6$
Let $y'' = 0$.

$6x - 6 = 0$
$x = 1$

Substitute $x = 1$ in the original equation.

$(y) = 1^3 - 3(1)^2 + 4$
$= 2$

At $x = 1$, $y'' = 0$. When $0 < x < 2$, $y' < 0$.

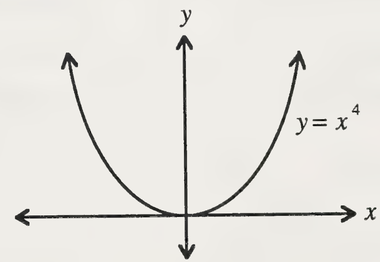
The slope remains negative and there is no direction change. Therefore, $(1, 2)$ is a point of inflection.

Example 5

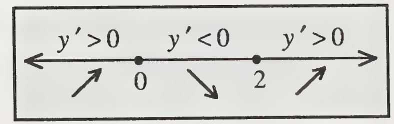
Is $(0, 0)$ a point of inflection of the curve defined by $y = x^4$?

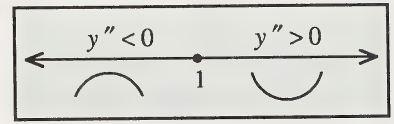
Solution:

If $y = x^4$, then $y' = 4x^3$ and $y'' = 12x^2$. At $x = 0$, $y'' = 0$; thus, it is possible that $(0, 0)$ is a point of inflection. However, the multiplicity of $x^4 = 0$ is an even number and 0 is the only root. Therefore, this curve should just touch the *x*-axis. For points in the neigbourhood of $(0, 0)$, when $x < 0$, $y' < 0$; and when $x > 0$, $y' > 0$. Therefore, the point $(0, 0)$ is not a point of inflection of the curve defined by $y = x^4$.



Look at some examples to demonstrate applications of this test.





Example 6

Determine the local extremes of $y = x^3 - x^2 - x + 5$, and state whether each extreme is a local maximum value or a local minimum value.

Solution:

The extremes are associated with the zeros of the first derivative.

$$
\begin{aligned}
y' &= 3x^2 - 2x - 1 \qquad (\text{Let } y' = 0.)\\
3x^2 - 2x - 1 &= 0\\
(3x+1)(x-1) &= 0\\
x = -\frac{1}{3} &\text{ or } x = 1
\end{aligned}
$$

The local maximum value and the local minimum values are found by substitution.

Substitute $x = -\frac{1}{3}$ in the original equation.

$$
\begin{aligned}
y &= \left(-\frac{1}{3}\right)^3 - \left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right) + 5\\
&= \frac{-1-3+9+135}{27}\\
&= \frac{140}{27}
\end{aligned}
$$

Substitute $x = 1$ in the original equation.

$$
\begin{aligned}
y &= 1^3 - 1^2 - 1 + 5\\
&= 4
\end{aligned}
$$

It appears that $y = \frac{140}{27}$ is a local maximum value and $y = 4$ is a local minimum value.

Use the second derivative test to confirm this observation.

The second derivative is $y'' = 6x - 2$.

At $x = -\frac{1}{3}$, $y'' = 6x - 2 = 6\left(-\frac{1}{3}\right) - 2 = -4$.
Thus, at this point $y' = 0$ and $y'' < 0$. These are conditions for a local maximum. Therefore, $y = \frac{140}{27}$ is a local maximum.

At $x = 1$, $y'' = 6x - 2 = 6(1) - 2 = 4$.
When $x = 1$, $y' = 0$ and $y'' > 0$. Therefore, $y = 4$ is a local minimum value.

Example 7

Answer the questions given the following:

$$
\begin{aligned}
y &= (x+2)^3(1-x)\\
y' &= (x+2)^2(1-4x)\\
y'' &= -6(x+2)(1+2x)
\end{aligned}
$$

- Verify that the given derivatives are correct.

Solution:

(Note: The given derivatives are correct. However, you should always verify that information such as this is correct.)

- State the critical values of the function and its derivatives.

Solution:

When $x=-2$ or $x=1$, $y=0$; thus, $(-2,0)$ and $(1,0)$ are the x-intercepts of the graph.

When $x=-2$ or $x=\frac{1}{4}$, $y'=0$.

Substitute $x=-2$ in the original equation.

$$\begin{aligned} y &= (-2+2)^3[1-(-2)] \\ &= 0 \end{aligned}$$

Substitute $x=\frac{1}{4}$ in the original equation.

$$\begin{aligned} y &= \left(\frac{1}{4}+2\right)^3\left(1-\frac{1}{4}\right) \\ &= \frac{2187}{256} \end{aligned}$$

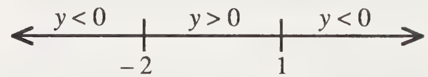
Therefore, $(-2,0)$ and $\left(\frac{1}{4},\frac{2187}{256}\right)$ are points where $y'=0$.

When $x=-2$ or $x=-\frac{1}{2}$, $y''=0$.

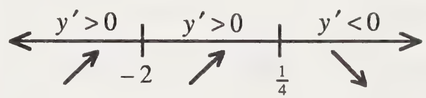
At $x=-2$, $y=0$.

At $x=-\frac{1}{2}$, $y=\frac{81}{16}$.

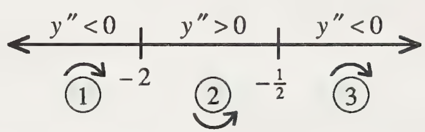
Therefore, $(-2,0)$ and $\left(-\frac{1}{2},\frac{81}{16}\right)$ are points where $y''=0$.



$y=(x+2)^3(1-x)$



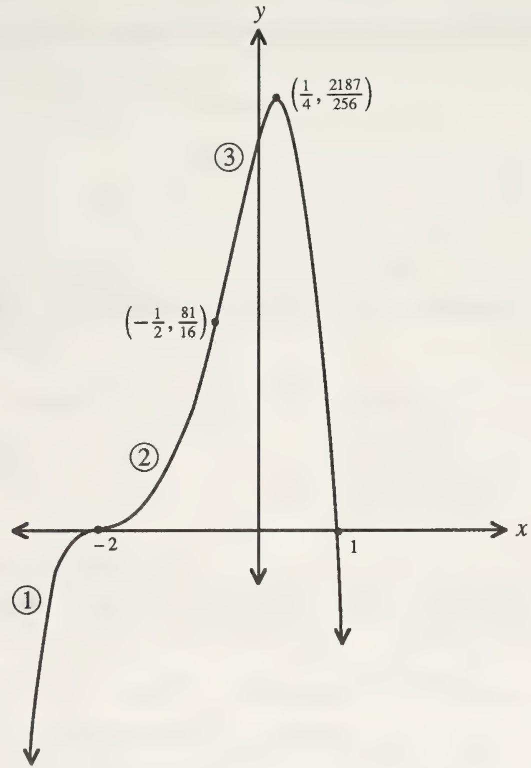
$y'=(x+2)^2(1-4x)$



$y''=-6(x+2)(1+2x)$

In expanded form the function is $y=-x^4-5x^3-6x^2+4x+8$ and the leading term is $-x^4$.

As $x\to\pm\infty$, $y\to-\infty$.



Study this example carefully before you go to the exercise.

Now try the following questions.

1. Determine the first derivative and its critical values for each of the following. Identify the intervals where the function is increasing and where it is decreasing.

   a. $y = x^2 + 4x + 8$

   b. $y = x^3 + 12x^2 + 45x + 16$

2. State the local maximum and the local minimum values of $f(x) = x^3 - 3x + 4$. Use the second derivative test to determine which are local maxima and which are local minima.

3. Show that the function given by $y = x^3 - 3x^2 + 4x$ has no maxima or minima for real values of x.

4. Show that the function $y = x - 3 + \frac{4}{x}, x \neq 0$ has its local minimum greater than its local maximum.

5. Show that the graph of $y = x^3 + x^2 - 5x + 3$ is tangent to the x-axis, and find the coordinates of the tangent point.

6. Show that the function $y = x + \frac{4}{x}, x \neq 0$ has a maximum and a minimum. What happens to the values of y when $x > 0$ and when $x < 0$? Use these results to show that y cannot have any value in the range $-4 < y < 4$.

7. Determine the coordinates of the points of inflection of the following graphs.

   a. $y = x^3 - 6x + 2$

   b. $y = 2x^4 + 8x^3$

   c. $y = x(x+3)^{\frac{1}{2}}$

8. Determine the intervals where the curve $y = 2x^3 - 15x^2 + 10x - 8$ is concave up or concave down.

9. Use the procedure outlined in Example 7 to draw the graphs of the following function and their derivatives. Graph paper is provided in **Appendix B**.

   a. $y = x^4 - 2x^2 - 8$
      $y' = 4x^3 - 4x$
      $y'' = 12x^2 - 4$

   b. $y = (x-2)^2(x+2)$
      $y' = (x-2)(3x+2)$
      $y'' = 6x - 4$

For solutions to **Activity 2**, turn to **Appendix A, Topic 2**.

Activity 3

Determine the concavity, the position, and the direction of the graph of a polynomial function.

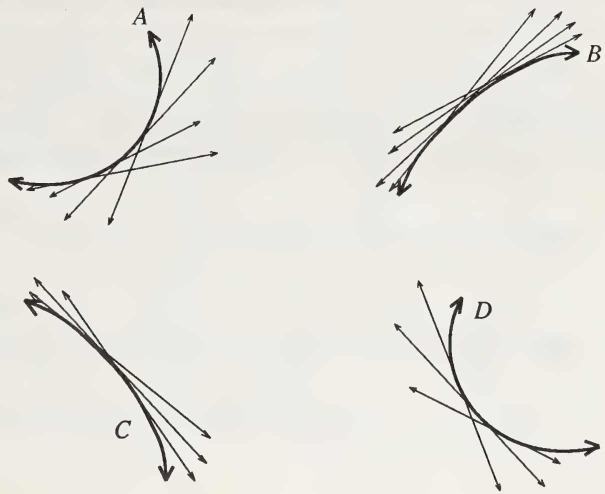
Often when you see a function, you wonder whether the curve will open up or down. The second derivative will show that the curve opens up or down.

If the second derivative of the function is negative, the slope is decreasing and the curve is **concave down** (opens down). If the second derivative is positive, the slope is increasing and the curve is **concave up** (opens up).

By the definition of maximum and minimum values at a point, a curve is concave down at a local maximum point and concave up at a local minimum point.

A local minimum function value occurs at an upward concavity where the first derivative has a value of zero. If the x-value which makes the first derivative equal to zero is substituted into the original function, then the minimum value of the original function is obtained. A local maximum function value occurs at a downward concavity where the first derivative has a value of zero. If the x-value which makes the first derivative equal to zero is substituted into the original function, then the maximum value of the original function is determined.

Study the slopes of the following figures. Note what happens as they get steeper and steeper. Also note at the difference between the slopes that are negative from those that are positive. Some of the figures are concave up while others are concave down.



A and D are concave up while B and C are concave down.

The second derivative of a function is very important when idetifying the bend of the curve. For example, if you know that the slope is positive, then the curve can be like A or B. Both A and B have a positive slope; yet A is concave up and B is concave down.

The second derivative for A is positive while the second derivative for B is negative. This shows the difference and indicates the direction of concavity.

If you have a difficulty when the slope is negative, remember that -2 is less then -1, but a line with slope of -2 is steeper than a line with a slope of -1.

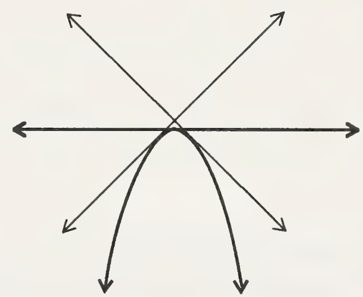
C has a decreasing slope, and the second derivative is negative. As negative numbers approach zero, they are increasing in magnitude; hence, D has an increasing slope and the second derivative is positive.

Therefore, if the second derivative is positive, the slope is increasing and the curve of the function is concave up. This results in a minimum function value.

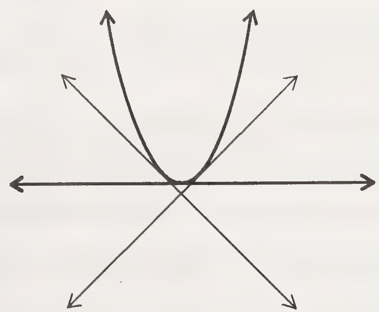
On the other hand if the second derivative is negative, the slope is decreasing and the curve is concave down. This results in a maximum function value.

The following illustration might give a better understanding of concavity.

- If $f''(x) < 0$, the curve is concave down (local maximum function value).



- If $f''(x) > 0$, the curve is concave up (local minimum function value).



In summary study the following statements.

- If $f'(x) = 0$ and $f''(x) > 0$, then $f(x)$ has a local maximum value.

- If $f'(x) = 0$ and $f''(x) < 0$, then $f(x)$ has a local maximum value.

Therefore, if the second derivative is positive in a region, the slope is increasing. You can say the curve is concave up. If a zero of the first derivative occurs in such an interval, it is possible to calculate the minimum function value. On the other hand if the second derivative is negative over an interval, the curve is concave down. Any zero of the first derivative in such an interval can be used to calculate a maximum function value.

Example 8

Determine the intervals where the curve defined by $f(x) = x^3 - 2x^2 - 5x + 6$ is concave up or concave down.

Solution:

$$y = x^3 - 2x^2 - 5x + 6$$
$$y' = 3x^2 - 4x - 5$$
$$y'' = 6x - 4$$

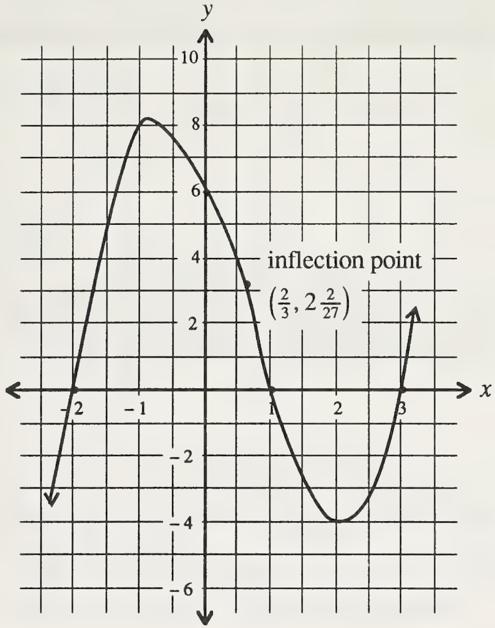
Let $6x - 4 = 0$.

$$x = \frac{2}{3}$$

$$x < \frac{2}{3} \rightarrow y'' < 0 \rightarrow \text{concave down}$$
$$x > \frac{2}{3} \rightarrow y'' > 0 \rightarrow \text{concave up}$$

The curve is concave up when x is greater than $\frac{2}{3}$. The curve is concave down when x is less than $\frac{2}{3}$.

At $x = \frac{2}{3}$, $y = 2\frac{2}{27}$. This represents the inflection point location since $y'' = 0$. Note that y'' changes sign as the function passes through the point $\left(\frac{2}{3}, 2\frac{2}{27}\right)$.



Now try some questions that deal with concavity.

1. Determine the regions where the curve $y = x^3 - 2x^2 - 7x - 4$ is concave up and concave down, and then draw the curve. Graph paper is provided in **Appendix B**.

2. The function $y = x^3 - 3x^2 + 4$ is concave up in one region and concave down in another. State where these concavities occur.

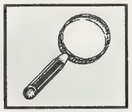
3. Sketch a curve $f(x)$ that is concave up for $x < 0$, concave down for $x > 0$, and the x-intercepts are -4, 0, and 4. What is the point of inflection?

For solutions to **Activity 3**, turn to **Appendix A, Topic 2**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

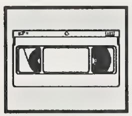
} You may decide to do both.



Extra Help

Do **Part A** or **Part B**.

Part A



If you have access to a videocassette recorder (VCR), view the video titled **Problems and Graph Sketching**. This video is program 5 in the *Catch 31*[1] series. Do the exercises at the end of this section after you have viewed the video program.

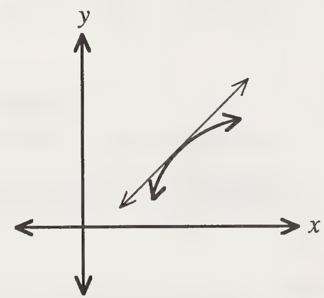
Part B

When you are sketching a polynomial function $f(x)$, the value of $\frac{dy}{dx}$ at a given point on the curve can show a number of things about the curve at that point.

Here is a summary which illustrates by use of a graph the information that can be gathered at this given point.

You will consider the following cases. $\left(\text{Remember: } f' = \frac{dy}{dx} = D_x y\right)$

- $D_x y$ is positive.



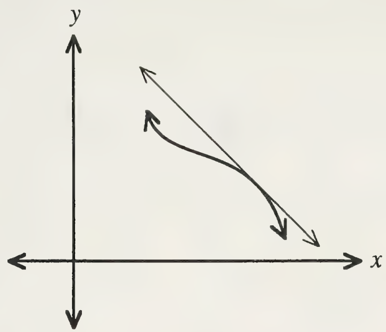
The slope of the tangent is positive.

The drawing shows that when x is increasing, y is increasing.

If $D_x y$ is positive, $f(x)$ is increasing.

[1] *Catch 31* is a title of ACCESS Network

- $D_x y$ is negative.



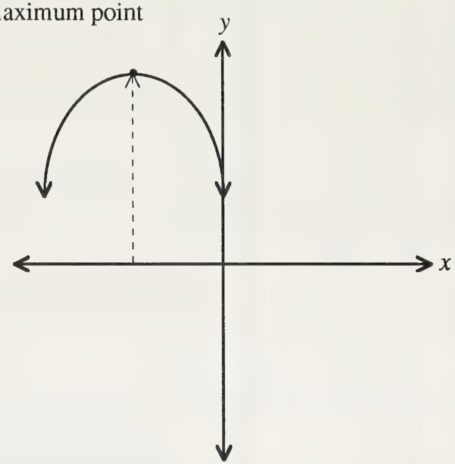
The slope of the tangent is negative.

The drawing shows that when x is increasing, y is decreasing.

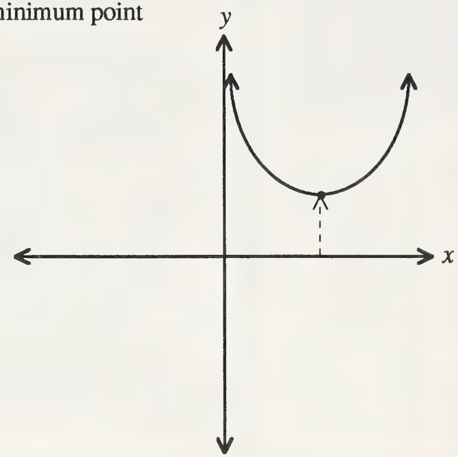
If $D_x y$ is negative, $f(x)$ is decreasing.

- $D_x y$ equals zero.

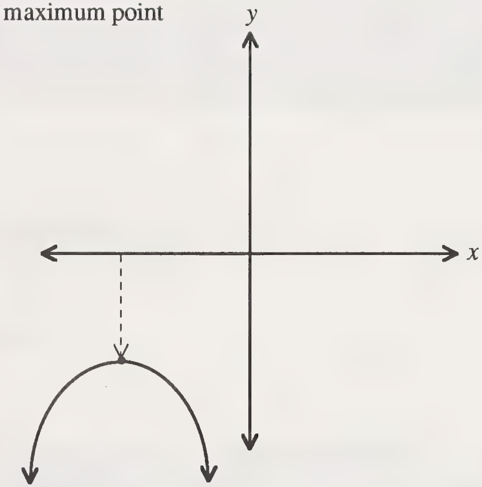
maximum point



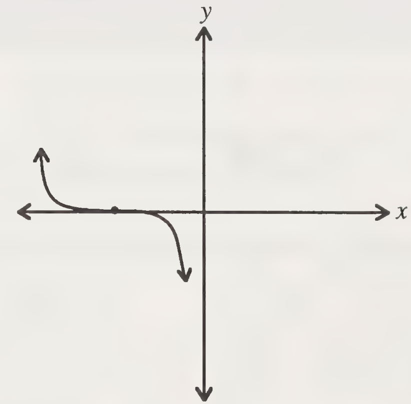
minimum point



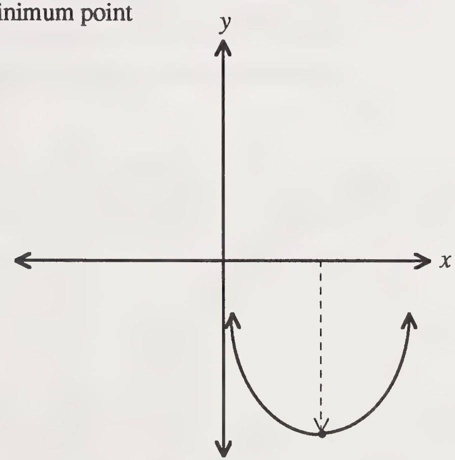
maximum point



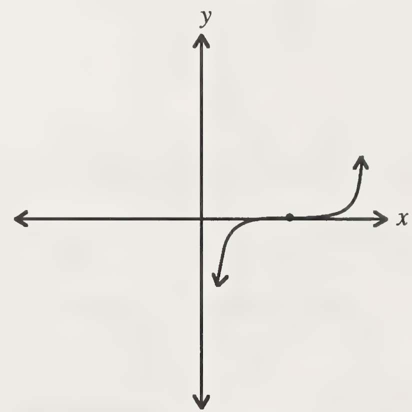
horizontal point of inflection (negative slope)



minimum point

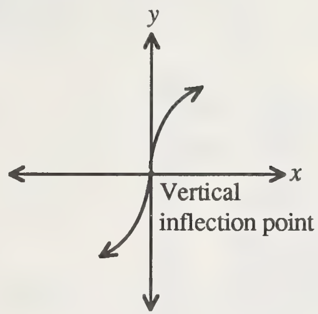


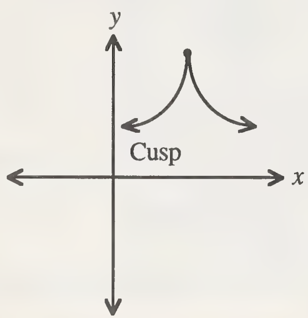
horizontal point of inflection (positive slope)

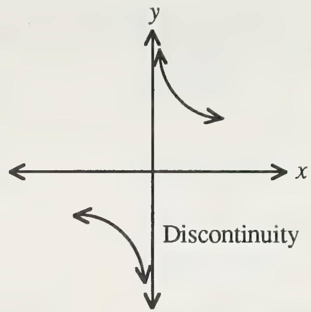


If $D_x y = 0$, the point is a maximum point, a minimum point, or a horizontal inflection point. Some texts refer to maximum points, minimum points, and those inflection points where $D_x y = 0$ as **stationary points**. The first derivative of a function at an inflection point is not necessarily zero. If it is zero, then the point is a stationary point.

- $D_x y$ does not exist.







If $D_x y$ is not defined, then the following may occur:

- The curve will be **parallel to the *y*-axis.**
- The curve will be **discontinuous** at this point.
- This point will be a **vertical inflection point**.
- The curve will form a **cusp** at this point.

These facts revealed by the first derivative are very useful in graph sketching. To sketch the graph of a function or relation means to draw the curve of the relation without actually plotting a detailed table of values of coordinates. To get the required curve, make use of the graph facts which its equation reveals. You should have already done some graph sketching in your mathematics study. You have already used the zeros of the first derivative and the sign of the second derivative to find maxima and minima as an aid in graph sketching.

A cusp is a peak resulting from an instantaneous change in the curvature of the function.

Example 9

Sketch the graph of the function $f(x) = x^3 - 9x^2 + 24x - 8$.

Solution:

To find the y-intercept, let $x = 0$.

$$y = (0)^3 - 9(0)^2 + 24(0) - 8$$
$$y = -8$$

Find the first derivative and the turning points. The first drivative equals zero at the turning points.

$$\frac{dy}{dx} = 3x^2 - 18x + 24 = 0$$
$$3(x-2)(x-4) = 0$$
$$x = 2 \text{ or } x = 4$$

Substitute $x = 2$ in the original equation.

$$y = (2)^3 - 9(2)^2 + 24(2) - 8$$
$$= 12$$

Substitute $x = 4$ in the original equation.

$$y = (4)^3 - 9(4)^2 + 24(4) - 8$$
$$= 8$$

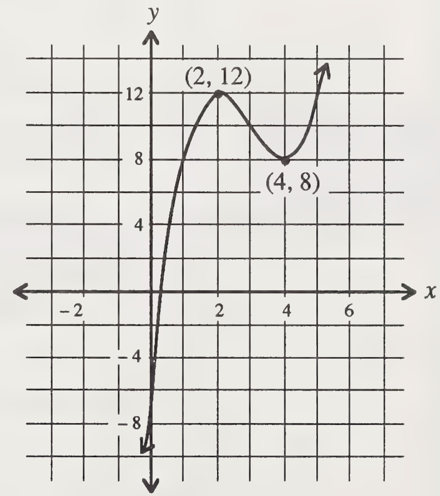
Therefore, the turning points are $(2, 12)$ and $(4, 8)$.

Find the intervals of x over which $\frac{dy}{dx} > 0$ and $\frac{dy}{dx} < 0$.

$\frac{dy}{dx} > 0$ for $x > 4$ and $x < 2$.

$\frac{dy}{dx} < 0$ for $2 < x < 4$.

Now that you have an idea of what the curve should look like, go ahead and sketch it.



Now try some of the following exercises.

1. Draw the curve of $y=(x-2)(x+1)(x+3)$. Graph paper is provided in **Appendix B**.

2. Draw the curve of $f(x)=x^3+3x^2-24x+28$. Graph paper is provided in **Appendix B**.

For solutions to **Extra Help**, turn to **Appendix A, Topic 2**.

Extensions

An added advantage when drawing the graph of a function is to test for symmetry. A function does not always reveal itself as to whether it is asymptotic or symmetric. Tests have to be conducted to determine the shape, curvature, and smoothness of the curve. Some functions are also discontinuous, and tests again have to be conducted to find out where the discontinuity exists.

Test for Symmetry

To determine if a graph is symmetric with respect to the different axes or the origin, look at the following procedure.

The graph of a relation is symmetric with respect to the following:

- the *x*-axis if the point $P_1(x,-y)$ is on the graph whenever the point $P(x,y)$ is on the graph
- the *y*-axis if the point $P_2(-x,y)$ is on the graph whenever the point $P(x,y)$ is on the graph
- the origin if the point $P_3(-x,-y)$ is on the graph whenever the point $P(x,y)$ is on the graph

The following procedures for symmetry are simple to apply and should be remembered. They are immediate consequents of the definitions given previously.

The graph of a relation is symmetric with respect to the following:

- the x-axis if the defining equation (or other defining conditions) of the relation remain unchanged when y is replaced by $(-y)$
- the y-axis if the defining conditions of the relation remains unchanged when x is replaced by $(-x)$
- the origin if the defining conditions of the relation remain unchanged when both x and y are replaced by $(-x)$ and $(-y)$ respectively

The following example shows the check for symmetry with respect to the origin.

Example 10

Determine if the graph of $y=\frac{1}{5}\left(x^5-4x^3\right)$ is symmetric with respect to the origin.

Solution:

If x is replaced by $-x$ and y is replaced by $-y$, then the following occurs.

$$-y=\frac{1}{5}\left[(-x)^5-4(-x)^3\right]$$
$$-y=\frac{1}{5}\left(-x^5+4x^3\right)\Leftrightarrow y=\frac{1}{5}\left(x^5-4x^3\right)$$

This is the test for symmetry with respect to the origin. The graph is symmetric with respect to the origin.

Test for Discontinuity

You will now study the function $y=|x|$.

For $y=|x|$ you know the following from the definition:

- $y=x$ for $x>0$ and $\frac{dy}{dx}=1$ for $x>0$.
- $y=-x$ for $x<0$ and $\frac{dy}{dx}=-1$ for $x<0$.

However, when $x=0$, $\frac{dy}{dx}$ does not exist (it is undefined) because $\frac{dy}{dx}$ is a limit. The limit does not exist when $x=0$.

For a limit to exist you must be able to approach it from the left and from the right, and still arrive at the same value.

Here, $\frac{dy}{dx}$ remains at -1 when you approach (0, 0) from the left, and $\frac{dy}{dx}$ remains at 1 when you approach (0, 0) from the right.

Test for Asymptotes

Consider the case for the function $y=\frac{12}{x}$ or $xy=12$.

Note that the limit $\lim\limits_{x\to 0}\left(\frac{12}{x}\right)$ is undefined.

This means that a vertical asymptote at $x=0$ exists.

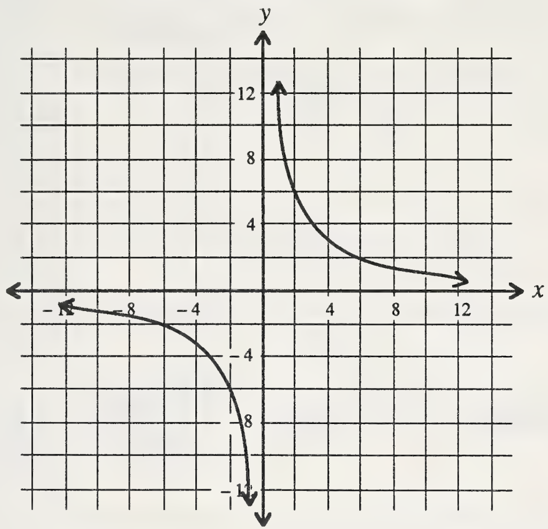
Now consider $\lim\limits_{x\to\infty}\left(\frac{12}{x}\right)=0$.

This suggests that a horizontal asymptote exists at $y = 0$.

Since $y = \frac{12}{x}$, then $D_x y = \frac{-12}{x^2}$.

Thus, $D_x y$ is always negative which tells you that the function is decreasing. $D_x y$ is not defined for $x = 0$. The tangents to the curve slope downward to the right. Note that each axis is an asymptote.

Using the following graph of $y = \frac{12}{x}$, explain why the expression $\frac{-12}{x^2}$ is negative for all nonzero x.



Now try the following exercise.

Explain how to draw each of the following, and then draw them.

1. $y = \dfrac{3}{(x-2)^2}$

2. $y^2 = 4x^3$

3. $y = \dfrac{9x}{x^2+1}$

For solutions to **Extensions**, turn to **Appendix A, Topic 2.**

Unit Summary

What You Have Learned

Upon completing this unit you should be able to apply the following rules for finding derivatives.

- $D_x[f(x) \pm g(x)] = D_x f(x) \pm D_x g(x)$
- power rule

  If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$.
- product rule

  If $y = u \bullet v$, then $D_x y = u\, D_x v + v\, D_x u$.
- chain rule

  If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$.

Unit Summary

- quotient rule

  If $y = \frac{u}{v}$, then $D_x y = \frac{v D_x u - u D_x v}{v^2}$.

You should also be able to draw the graph of a polynomial function by making use of critical points such as the *y*- and *x*-intercepts, minimum and maximum points, and points of inflection.

Furthermore, you should be able to find the derivatives of relations.

You are now ready to complete the **Unit Assignment**.

Appendices

Appendix A
Solutions

Appendix B
Graphing Material

Appendix A
Solutions

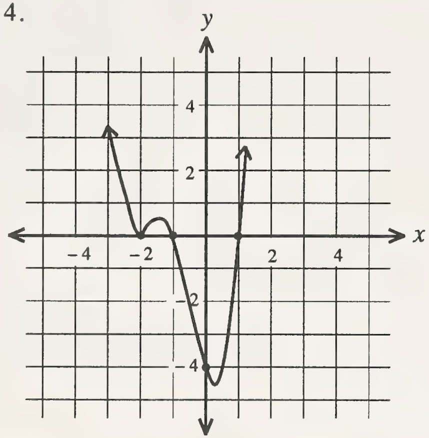
Review

1. a. $P(x)=(2x-1)(x-6)$
   The zeros are $\frac{1}{2}$ and 6.

   b. $P(x)=(x+1)^2(x-1)(x-2)$
   The zeros are -1, 1, and 2.

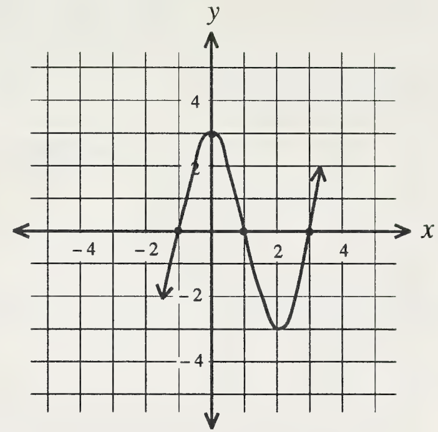
2. a. The zeros are -2, 1, and -1.
   The y-intercept is -4.

x	y
-2	0
1	0
-1	0
0	4



   b. $P(x)=(x-1)(x+1)(x-3)$
   The zeros are 1, -1, and 3.
   The y-intercept is 3.

x	y
1	0
-1	0
3	0
0	3



3. a. $(x-5)(x+4)=0$

$$x-5=0 \text{ or } x+4=0$$
$$x=5 \qquad x=-4$$

   b. $(2x+1)(2x-5)=0$

$$2x+1=0 \text{ or } 2x-5=0$$
$$x=-\frac{1}{2} \qquad x=\frac{5}{2}$$

c. $x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-3)}}{2(3)}$

$x = \frac{1 \pm \sqrt{37}}{6}$

$x = \frac{1+\sqrt{37}}{6}$ or $x = \frac{1-\sqrt{37}}{6}$

4. a. $y - y_1 = m(x - x_1)$

$y - (-6) = -3(x-1)$

$y + 6 = -3x + 3$

$3x + y + 3 = 0$

You can also use $y = mx + b$ to find the equation of a line.

$y = mx + b$

$-6 = -3(1) + b$

$-6 = -3 + b$

$-3 = b$

Therefore, the equation of the line is $y = -3x - 3$ or $3x + y + 3 = 0$.

b. $m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{3-1}{1-(-2)}$

$m = \frac{2}{3}$

$y - y_1 = m(x - x_1)$

$y - 3 = \frac{2}{3}(x-1)$ (Multiply by 3.)

$3y - 9 = 2(x-1)$

$-2x + 3y - 7 = 0$ or $2x - 3y + 7 = 0$

c. The point of the y-intercept is $(0, 6)$.

$y - y_1 = m(x - x_1)$

$y - 6 = \frac{1}{4}(x-0)$ (Multiply by -4.)

$-4y + 24 = -1x$

$x - 4y + 24 = 0$

You can also use $y = mx + b$.

$y = \frac{1}{4}x + 6$ (Multiply by 4.)

$4y = 1x + 24$

$-x + 4y - 24 = 0$ or $x - 4y + 24 = 0$

5. a. $y = -4x + 2$

$\therefore m = -4$

The slope of the perpendicular is $\frac{1}{4}$ (negative reciprocal).

b. $2x + 3y - 6 = 0$

$$\frac{3y}{3} = \frac{-2x}{3} + \frac{6}{3}$$

$$y = -\frac{2}{3}x + 2$$

$$m = -\frac{2}{3}$$

Therefore, the slope of the parallel line is $-\frac{2}{3}$. (Slopes of parallel lines are equal.)

6. a.

$$\begin{array}{rrl} & 2x + y - 6 = 0 & (1) \\ & x - y + 3 = 0 & (2) \\ \hline (1) + (2): & 3x - 3 = 0 & \\ & 3x = 3 & \\ & x = 1 & \end{array}$$

When you substitute $x = 1$ in (2), $y = 4$.

The solution is $(1, 4)$.

You may use the addition or elimination method.

b.

$$\begin{array}{rl} 3x + 2y = -4 & (1) \\ 4x - 3y = -11 & (2) \end{array}$$

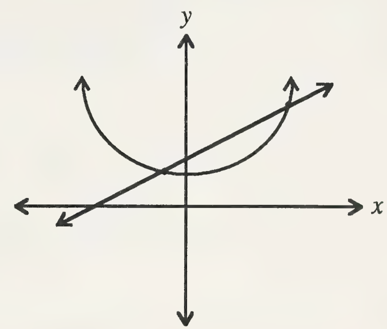
$$\begin{array}{rrl} 3 \times (1): & 9x + 6y = -12 & (3) \\ 2 \times (2): & 8x - 6y = -22 & (4) \\ \hline (3) + (4): & 17x = -34 & \\ & x = -2 & \end{array}$$

Substitute $x = -2$, in (1).

$$-6 + 2y = -4$$
$$2y = 2$$
$$y = 1$$

The solution is $(-2, 1)$.

c. This question may be a little different from that you are used to. The following diagram shows two answers.



$$y = x^2 + 1 \quad (1)$$
$$y = x + 3 \quad (2)$$

Since $y = y$, then equate (1) and (2).

$$x^2 + 1 = x + 3$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$(x-2) = 0 \text{ or } x+1 = 0$$
$$x = 2 \qquad x = -1$$

If $x = 2$, then $y = 5$.
If $x = -1$, then $y = 2$.
The points of intersection are $(2, 5)$ and $(-1, 2)$.

Exploring Topic 1

Activity 1

Determine the derivative of the sum or difference of two functions.

1. $\frac{dy}{dx} = (2)(4)x^3 - (3)(3)x^2 - 5$
   $= 8x^3 - 9x^2 - 5$

2. $\frac{dy}{dx} = (5)x^4 - (8)(4)x^3 + (3)(3)x^2$
   $= 5x^4 - 32x^3 + 9x^2$

3. $\frac{dy}{dx} = (5)(4)x^3 - \left(\frac{1}{2}\right)(3)x^2 + 0$
   $= 20x^3 - \frac{3}{2}x^2$

4. $\frac{dy}{dx} = \left(\frac{1}{2}\right)(3)x^2 - (2)(2)x + 1$
   $= \frac{3}{2}x^2 - 4x + 1$

Activity 2

Use the power rule and its applications.

1. $\frac{dy}{dx} = 0$

2. $\frac{dy}{dx} = (5)(2)x$
   $= 10x$

3. $\frac{dy}{dx} = (3)(4)x^3$
   $= 12x^3$

4. $\dfrac{dy}{dx}=0$

5. $\dfrac{dy}{dx}=0-5+(12)(2)x$

$=-5+24x$

6. $\dfrac{dy}{dx}=0+(6)(2)x+4\left(x^3\right)$

$=12x+4x^3$

7. $\dfrac{dy}{dx}=\left(\dfrac{1}{2}\right)(5)x^4-6(4)x^3+\dfrac{1}{3}+0$

$=\dfrac{5}{2}x^4-24x^3+\dfrac{1}{3}$

8. $\dfrac{dy}{dx}=\left(\dfrac{1}{3}\right)(4)x^3-\left(\dfrac{1}{2}\right)(3)x^2-5+0$

$=\dfrac{4}{3}x^3-\dfrac{3}{2}x^2-5$

9. $\dfrac{dy}{dx}=\left(\dfrac{1}{2}\right)x^{\frac{1}{2}-1}-\left(\dfrac{5}{6}\right)x^{\frac{5}{6}-1}$

$=\dfrac{1}{2}x^{-\frac{1}{2}}-\dfrac{5}{6}x^{-\frac{1}{6}}$

$=\dfrac{1}{2x^{\frac{1}{2}}}-\dfrac{5}{6x^{\frac{1}{6}}}$

10. $\dfrac{dy}{dx}=\dfrac{2}{3}\left(\dfrac{1}{3}\right)x^{\frac{1}{3}-1}-2\left(\dfrac{1}{2}\right)x^{\frac{1}{2}-1}$

$=\dfrac{2}{9}x^{-\frac{2}{3}}-x^{-\frac{1}{2}}$

$=\dfrac{2}{9x^{\frac{2}{3}}}-\dfrac{1}{x^{\frac{1}{2}}}$

11. $\dfrac{dy}{dx}=\left(\dfrac{1}{5}\right)\left(-\dfrac{1}{5}\right)x^{-\frac{1}{5}-1}-(4)\left(-\dfrac{1}{4}\right)x^{-\frac{1}{4}-1}$

$=-\dfrac{1}{25}x^{-\frac{6}{5}}+x^{-\frac{5}{4}}$

$=-\dfrac{1}{25x^{\frac{6}{5}}}+\dfrac{1}{x^{\frac{5}{4}}}$

12. $y=3x^{-\frac{1}{3}}+\dfrac{4}{3}x^{-\frac{1}{4}}$

$\dfrac{dy}{dx}=(3)\left(-\dfrac{1}{3}\right)x^{-\frac{1}{3}-1}+\left(\dfrac{4}{3}\right)\left(-\dfrac{1}{4}\right)x^{-\frac{1}{4}-1}$

$=-x^{-\frac{4}{3}}-\dfrac{1}{3}x^{-\frac{5}{4}}$

$=-\dfrac{1}{x^{\frac{4}{3}}}-\dfrac{1}{3x^{\frac{5}{4}}}$

Activity 3

Use the product rule and its applications.

1. $\frac{dy}{dx}=\left(x^2-3x+4\right)\frac{d\left(2x^2-x+1\right)}{dx}+\left(2x^2-x+1\right)\frac{d\left(x^2-3x+4\right)}{dx}$

$=\left(x^2-3x+4\right)(4x-1)+\left(2x^2-x+1\right)(2x-3)$

$=4x^3-12x^2+16x-x^2+3x-4+4x^3-2x^2+2x-6x^2+3x-3$ (You may choose to expand $\frac{dy}{dx}$.)

$=8x^3-21x^2+24x-7$

2. $D_x y=\left(x^2+4x-5\right)D_x\left(3x^2+x-1\right)+\left(3x^2+x-1\right)D_x\left(x^2+4x-5\right)$

$=\left(x^2+4x-5\right)(6x+1)+\left(3x^2+x-1\right)(2x+4)$

$=6x^3+24x^2-30x+x^2+4x-5+6x^3+2x^2-2x+12x^2+4x-4$

$=12x^3+39x^2-24x-9$

3. $D_x y=x^{-2}D_x(x+3)+(x+3)D_x\left(x^{-2}\right)$

$=x^{-2}(1)+(x+3)(-2)x^{-3}$

$=x^{-2}-2(x+3)x^{-3}$

4. $D_x y = x^{-5} D_x (x-7) + (x-7) D_x \left(x^{-5}\right)$

$= x^{-5}(1) + (x-7)(-5)x^{-6}$

$= x^{-5} - 5(x-7)x^{-6}$

5. $D_x y = (x-3)^{-2} D_x (x+1) + (x+1) D_x (x-3)^{-2}$

$= (x-3)^{-2}(1) + (x+1)(-2)(x-3)^{-3}(1)$

$= (x-3)^{-2} - 2(x+1)(x-3)^{-3}$

6. $D_x y = (x+4)^{-3} D_x (x-2) + (x-2) D_x (x+4)^{-3}$

$= (x+4)^{-3}(1) + (x-2)(-3)(x+4)^{-4}(1)$

$= (x+4)^{-3} - 3(x-2)(x+4)^{-4}$

7. $D_x y = (x-5) D_x \sqrt{x+2} + \sqrt{x+2}\, D_x (x-5)$

$= (x-5)\left(\frac{1}{2}\right)(x+2)^{-\frac{1}{2}}(1) + (x+2)^{\frac{1}{2}}(1)$

$= \frac{1}{2}(x-5)(x+2)^{-\frac{1}{2}} + (x+2)^{\frac{1}{2}}$

$= \frac{1}{2}(x+2)^{-\frac{1}{2}}[(x-5) + 2(x+2)]$

$= \frac{1}{2}(x+2)^{-\frac{1}{2}}(3x-1)$

8. $D_x y = (x-3)^{\frac{1}{2}} D_x (x+1) + (x+1) D_x (x-3)^{\frac{1}{2}}$

$= (x-3)^{\frac{1}{2}}(1) + (x+1)\left(\frac{1}{2}\right)(x-3)^{-\frac{1}{2}}(1)$

$= (x-3)^{\frac{1}{2}} + \frac{1}{2}(x+1)(x-3)^{-\frac{1}{2}}$

$= \frac{1}{2}(x-3)^{-\frac{1}{2}}[2(x-3) + (x+1)]$

$= \frac{1}{2}(x-3)^{-\frac{1}{2}}(3x-5)$

9. $D_x y = (x-2)^{\frac{1}{2}} D_x (x+1)^{\frac{1}{2}} + (x+1)^{\frac{1}{2}} D_x (x-2)^{\frac{1}{2}}$

$= (x-2)^{\frac{1}{2}}\left(\frac{1}{2}\right)(x+1)^{-\frac{1}{2}}(1) + (x+1)^{\frac{1}{2}}\left(\frac{1}{2}\right)(x-2)^{-\frac{1}{2}}(1)$

$= \frac{1}{2}(x+1)^{-\frac{1}{2}}(x-2)^{-\frac{1}{2}}(x-2+x+1)$

$= \dfrac{2x-1}{2(x+1)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}}$

10. $D_x y = (x-3)^{\frac{1}{2}} D_x (x-5)^{\frac{1}{2}} + (x-5)^{\frac{1}{2}} D_x (x-3)^{\frac{1}{2}}$

$= (x-3)^{\frac{1}{2}}\left(\frac{1}{2}\right)(x-5)^{-\frac{1}{2}}(1) + (x-5)^{\frac{1}{2}}\left(\frac{1}{2}\right)(x-3)^{-\frac{1}{2}}(1)$

$= \frac{1}{2}(x-5)^{-\frac{1}{2}}(x-3)^{-\frac{1}{2}}(x-3+x-5)$

$= \dfrac{2x-8}{2(x-5)^{\frac{1}{2}}(x-3)^{\frac{1}{2}}}$

Activity 4

Use the chain rule and its applications.

1. $y=(5x-3)^6$

$$\frac{dy}{dx}=6(5x-3)^5\,\frac{d(5x-3)}{dx}$$
$$=6(5x-3)^5(5)$$
$$=30(5x-3)^5$$

OR Let $u=5x-3$.

$$D_x u=5$$
$$y=u^6$$
$$\frac{dy}{dx}=\frac{dy}{du}\bullet\frac{du}{dx}$$
$$=6u^5\bullet 5$$
$$=30u^5$$
$$=30(5x-3)^5$$

2. $y=(6x+7)^7$

$$\frac{dy}{dx}=7(6x+7)^6\,\frac{d(6x+7)}{dx}$$
$$=42(6x+7)^6$$

OR Let $u=6x+7$.

$$D_x u=6$$
$$y=u^7$$
$$D_u y=7u^6$$
$$D_x y=D_u y\bullet D_x u$$
$$=7u^6(6)$$
$$=42(6x+7)^6$$

3. $y=(x-5)^{-1}$

$$\frac{dy}{dx}=(-1)(x-5)^{-2}$$
$$=-\frac{1}{(x-5)^2}$$

4. $y=4\left(x^2+2\right)^{-1}$

$$\frac{dy}{dx}=4(-1)\left(x^2+2\right)^{-2}D_x\left(x^2+2\right)$$
$$=-4\left(x^2+2\right)^{-2}(2x)$$
$$=-8x\left(x^2+2\right)^{-2}$$

OR Let $u=x^2+2$.

$$D_x u=2x$$
$$y=4u^{-1}$$
$$D_u y=-4u^{-2}$$
$$D_x y=-4u^{-2}\bullet 2x$$
$$=-8xu^{-2}$$
$$=-8x\left(x^2+2\right)^{-2}$$

5. $y=(3x-1)^{\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{1}{2}(3x-1)^{-\frac{1}{2}}(3)$$
$$=\frac{3}{2}(3x-1)^{-\frac{1}{2}}$$

6. $y=(4x+3)^{\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{1}{2}(4x+3)^{-\frac{1}{2}}(4)$$
$$=2(4x+3)^{-\frac{1}{2}}$$

7. $y=3(x-2)^{-\frac{1}{2}}$

$$\frac{dy}{dx}=(3)\left(-\frac{1}{2}\right)(x-2)^{-\frac{3}{2}}(1)$$
$$=-\frac{3}{2}(x-2)^{-\frac{3}{2}}$$

8. $y=5(2x+1)^{-\frac{1}{2}}$

$$\frac{dy}{dx}=(5)\left(-\frac{1}{2}\right)(2x+1)^{-\frac{3}{2}}(2)$$
$$=-5(2x+1)^{-\frac{3}{2}}$$

9. $y=\left(20-7x^2\right)^5$

$$\frac{dy}{dx}=5\left(20-7x^2\right)^4 D_x\left(20-7x^2\right)$$
$$=5\left(20-7x^2\right)^4(-14x)$$
$$=-70x\left(20-7x^2\right)^4$$

OR

Let $u=20-7x^2$.

$D_x u=-14x$

$y=u^5$

$D_u y=5u^4$

$$D_x y=D_u y \cdot D_x u$$
$$=5u^4(-14x)$$
$$=-70xu^4$$
$$=-70x\left(20-7x^2\right)^4$$

10. $y=3x^{-\frac{5}{3}}$

$$D_x y=(3)\left(-\frac{5}{3}\right)x^{-\frac{5}{3}-1}$$
$$=-5x^{-\frac{8}{3}}$$

Activity 5

Use the quotient rule and its applications.

1. $$\frac{dy}{dx}=\frac{(5x+7)D_x(3x-2)-(3x-2)D_x(5x+7)}{(5x+7)^2}$$
$$=\frac{(5x+7)(3)-(3x-2)(5)}{(5x+7)^2}$$
$$=\frac{15x+21-15x+10}{(5x+7)^2}$$
$$=\frac{31}{(5x+7)^2}$$

2. $$D_x y = \frac{\left(5+x+2x^2\right)D_x\left(2-3x+x^2\right)-\left(2-3x+x^2\right)D_x\left(5+x+2x^2\right)}{\left(5+x+2x^2\right)^2}$$

$$= \frac{\left(5+x+2x^2\right)(-3+2x)-\left(2-3x+x^2\right)(1+4x)}{\left(5+x+2x^2\right)^2}$$

$$= \frac{\left(-15-3x-6x^2+10x+2x^2+4x^3\right)-\left(2-3x+x^2+8x-12x^2+4x^3\right)}{\left(5+x+2x^2\right)^2}$$

$$= \frac{7x^2+2x-17}{\left(5+x+2x^2\right)^2}$$

3. $$D_x y = \frac{\left(3x-x^2\right)D_x(5x-1)-(5x-1)D_x\left(3x-x^2\right)}{\left(3x-x^2\right)^2}$$

$$= \frac{\left(3x-x^2\right)(5)-(5x-1)(3-2x)}{\left(3x-x^2\right)^2}$$

$$= \frac{15x-5x^2-15x+3+10x^2-2x}{\left(3x-x^2\right)^2}$$

$$= \frac{5x^2-2x+3}{\left(3x-x^2\right)^2}$$

4.

$$D_x y = \frac{\left(\sqrt{2x^2+1}\right) D_x x - (x) D_x \sqrt{2x^2+1}}{\left(2x^2+1\right)}$$

$$= \frac{\sqrt{2x^2+1} - (x)\left(\frac{1}{2}\right)\left(2x^2+1\right)^{-\frac{1}{2}}(4x)}{\left(2x^2+1\right)}$$

$$= \frac{\left(2x^2+1\right)^{\frac{1}{2}} - 2x^2\left(2x^2+1\right)^{-\frac{1}{2}}}{\left(2x^2+1\right)}$$

$$= \frac{\left(2x^2+1\right)^{-\frac{1}{2}}\left(\cancel{2x^2}+1-\cancel{2x^2}\right)}{\left(2x^2+1\right)}$$

$$= \frac{\left(2x^2+1\right)^{-\frac{1}{2}}}{\left(2x^2+1\right)^{1}}$$

$$= \frac{1}{\left(2x^2+1\right)^{\frac{3}{2}}}$$

5.

$$D_x y = \frac{\left(x^2-3\right)^{\frac{1}{2}} D_x (2x-3) - (2x-3) D_x \left(x^2-3\right)^{\frac{1}{2}}}{x^2-3}$$

$$= \frac{\left(x^2-3\right)^{\frac{1}{2}}(2) - (2x-3)\left(\frac{1}{2}\right)\left(x^2-3\right)^{-\frac{1}{2}}(2x)}{x^2-3}$$

$$= \frac{\left(x^2-3\right)^{\frac{1}{2}}(2) - x(2x-3)\left(x^2-3\right)^{-\frac{1}{2}}}{x^2-3}$$

$$= \frac{\left(x^2-3\right)^{-\frac{1}{2}}\left[\left(x^2-3\right)(2) - x(2x-3)\right]}{x^2-3}$$

$$= \frac{\left(x^2-3\right)^{-\frac{1}{2}}\left(2x^2-6-2x^2+3x\right)}{x^2-3}$$

$$= \frac{\left(x^2-3\right)^{-\frac{1}{2}}(3x-6)}{\left(x^2-3\right)}$$

$$= \frac{3x-6}{\left(x^2-3\right)^{\frac{3}{2}}}$$

6. $$D_x y = \frac{(3-x^2)^{\frac{1}{2}} D_x(-5x) - (-5x) D_x (3-x^2)^{\frac{1}{2}}}{(3-x^2)}$$

$$= \frac{(3-x^2)^{\frac{1}{2}}(-5) - (-5x)\left(\frac{1}{2}\right)(3-x^2)^{-\frac{1}{2}}(-2x)}{(3-x^2)}$$

$$= \frac{-5(3-x^2)^{\frac{1}{2}} - 5x^2(3-x^2)^{-\frac{1}{2}}}{(3-x^2)}$$

$$= \frac{(3-x^2)^{-\frac{1}{2}}\left[-5(3-x^2) - 5x^2\right]}{(3-x^2)}$$

$$= \frac{(3-x^2)^{-\frac{1}{2}}(-15+5x^2-5x^2)}{(3-x^2)}$$

$$= \frac{-15}{(3-x^2)^{\frac{3}{2}}}$$

Activity 6

Determine the derivatives of relations.

1. a.
$$3xy + 2x^2 + y^2 = 2$$
$$3xy' + 3y + 4x + 2yy' = 0$$
$$y' = \frac{-3y-4x}{3x+2y}$$

b.
$$y^3 - 3xy^2 - 2x^2y + x = 2$$
$$3y^2y' - 3x(2yy') - 3y^2 - 2x^2y' - 4xy + 1 = 0$$
$$\frac{3y^2 + 4xy - 1}{3y^2 - 6xy - 2x^2} = y'$$

c.
$$\frac{x^{\frac{1}{3}}}{a} - \frac{y^{\frac{1}{3}}}{a} = 1$$
$$x^{\frac{1}{3}} - y^{\frac{1}{3}} = a$$
$$\frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{3}y^{-\frac{2}{3}}y' = 0$$
$$x^{-\frac{2}{3}} - y^{-\frac{2}{3}}y' = 0$$
$$y' = \frac{-x^{-\frac{2}{3}}}{-y^{-\frac{2}{3}}}$$
$$= \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}$$

d. $$xy^2 = 2x - \frac{1}{x}$$

$$x(2yy') + y^2 = 2 + \frac{1}{x^2}$$

$$y' = \frac{\frac{2x^2+1}{x^2} - y^2}{2xy}$$

$$y' = \frac{2x^2 + 1 - x^2y^2}{2x^3y}$$

2. $$x^2 + 4xy - 2y = 3$$

$$2x + 4xy' + 4y - 2y' = 0$$

$$y' = \frac{-2x-4y}{4x-2}$$

$$= \frac{-x-2y}{2x-1}$$

Solve for y' at $P(1,1)$.

$$m = \frac{-3}{1}$$

Substitute $m = \frac{-3}{1}$ and $P(1,1)$ in the slope-point formula.

$$y - 1 = \frac{-3}{1}(x-1)$$

$$3x + y - 4 = 0$$

Therefore, the tangent to the graph is $3x + y - 4 = 0$.

Extra Help

1. $y = x^{10}$

   $D_x y = 10x^9$

2. $y = x^{-3}$

   $D_x y = -3x^{-4}$

3. $y = 3x^5$

   $D_x y = 15x^4$

4. $y = 2x^{-2}$

   $D_x y = -4x^{-3}$

5. $y = 3x^2 - 4x + 7$

   $D_x y = 6x - 4$

6. $y = (x-3)(x^2+5)$

$$D_x y = (x-3)D_x(x^2+5) + (x^2+5)D_x(x-3)$$

$$= (x-3)(2x) + (x^2+5)(1)$$

$$= 2x^2 - 6x + x^2 + 5$$

$$= 3x^2 - 6x + 5$$

7. $y=(7-2x)^6$

$$\begin{aligned} D_x y &= 6(7-2x)^5(-2) \\ &= -12(7-2x)^5 \end{aligned}$$

8. $y=\dfrac{2x-3}{3x+8}$

$$\begin{aligned} D_x y &= \frac{(3x+8)D_x(2x-3)-(2x-3)D_x(3x+8)}{(3x+8)^2} \\ &= \frac{(3x+8)(2)-(2x-3)(3)}{(3x+8)^2} \\ &= \frac{6x+16-6x+9}{(3x+8)^2} \\ &= \frac{25}{(3x+8)^2} \end{aligned}$$

Extensions

$y=x^n$, where $x \in R$ and $n \in N$.

Suppose that x changes by a small quantity Δx, and y changes by Δy.

$$\begin{aligned} (y+\Delta y) &= (x+\Delta x)^n \\ \Delta y &= (x+\Delta x)^n - y \\ &= (x+\Delta x)^n - x^n \\ &= (x+\Delta x - x)\left[(x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x + \cdots + x^{n-1}\right] \\ &= \Delta x\left[(x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x + \cdots + x^{n-1}\right] \end{aligned}$$

$$\therefore \frac{\Delta y}{\Delta x} = \left[(x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x + \cdots + x^{n-1}\right]$$

$$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left[(x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x + \cdots + x^{n-1}\right]$$

Since $\lim\limits_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$, you can state the following:

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0}\left[(x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x + \cdots + x^{n-1}\right] \\ &= \underbrace{x^{n-1} + x^{n-1} + \cdots + x^{n-1}}_{n \text{ terms}} \\ &= nx^{n-1} \end{aligned}$$

Exploring Topic 2

Activity 1

Apply the first derivative to determine coordinates of the turning points, and define and determine the critical values.

1. a. i. $f'(x)$ is correct.

ii. The $f(x)$-intercept is 0.
To find the x-intercepts, do the following:

$$-x^3 + 3x = 0$$
$$-x(x^2 - 3) = 0$$
$$-x = 0 \text{ or } x^2 - 3 = 0$$
$$x = 0 \qquad x = \pm\sqrt{3}$$

The critical values of $f(x)$ are 0 and $\pm\sqrt{3}$.

To find the critical values for $f'(x)$, do the following:

$$-3x^2 + 3 = 0$$
$$-3x^2 = -3$$
$$x^2 = 1$$
$$x = \pm 1$$

The critical values of $f'(x)$ are $x = \pm 1$. These critical values are used to locate the turning points.

At $x = 1, f(x) = 2$; thus, $(1, 2)$ is a turning point.

At $x = -1, f(x) = -2$; thus, $(-1, -2)$ is a turning point.

iii. The function is of the third degree with a negative controlling term. This means the graph will go in three directions and end up going down. This is the down-up-down characteristic of a negative cubic.

The following table is for $f(x)$.

Interval	Test Point	$f(x)$
$x < -\sqrt{3}$	-4	+ (above x-axis)
$-\sqrt{3} < x < 0$	-1	– (below x-axis)
$0 < x < \sqrt{3}$	1	+
$x > \sqrt{3}$	4	–

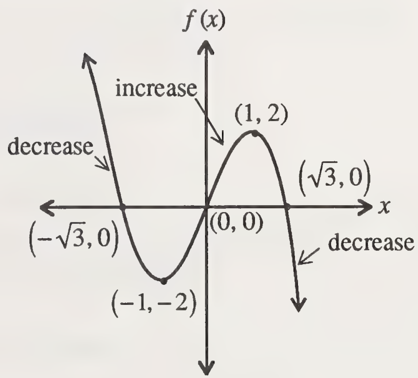
As $x \to \infty, f(x) \to -\infty$.

As $x \to -\infty, f(x) \to \infty$.

The following table is for $f'(x)$.

Interval	Test Point	$f'(x)$	Interpretation
$x<-1$	-3	$-$	decreasing
$-1<x<1$	0	$+$	increasing
$x>1$	3	$-$	decreasing

iv.



b. i. $g(x)=(x-2)^2(x+1)$

$$g'(x)=(x-2)^2(1)+(x+1)(2)(x-2)$$
$$=(x-2)(x-2+2x+2)$$
$$=(x-2)(3x)$$
$$=3x(x-2)$$

ii. The $g(x)$-intercept is 4.
To find the x-intercept, do as follows:

$(x+1)(x-2)^2=0$

$x+1=0$ or $(x-2)^2=0$

$x=-1$ $\quad$ $x=2$

Critical points for $g'(x)$ do as follows:

$3x(x-2)=0$

$3x=0$ or $x-2=0$

$x=0$ $\quad$ $x=2$

At $x=0$, $g(x)=4$; thus, $(0,4)$ is a critical point.

At $x=2$, $g(x)=0$; thus, $(2,0)$ is a critical point.

iii. The function is of the third degree, but it has multiplicities. For $(x-2)^2$ the curve will touch the x-axis at 2. The graph will go in three directions and end up going upwards. This is the classic up-down-up of a positive cubic.

The following table is for $g(x)$.

Interval	Test Point	$g(x)$
$x<-1$	-3	$-$ (below x-axis)
$-1<x<2$	0	$+$ (above x-axis)
$x>2$	3	$+$

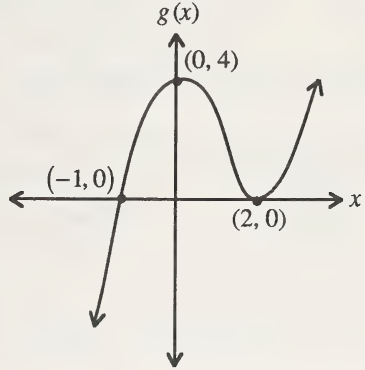
As $x \to \infty, g(x) \to \infty$.

As $x \to -\infty, g(x) \to -\infty$.

The following table is for $g'(x)$.

Interval	Test Point	$g'(x)$	Interpretation
$x<0$	-2	$+$	increasing
$0<x<2$	1	$-$	decreasing
$x>2$	4	$+$	increasing

iv.



2. $y=(x-1)^3(x+2)^2$

$$y'=(x-1)^3(2)(x+2)+(x+2)^2(3)(x-1)^2$$
$$=(x+2)(x-1)^2[(x-1)(2)+3(x+2)]$$
$$=(x+2)(x-1)^2(5x+4)$$

Critical values for y' are as follows:

$$(x+2)(x-1)^2(5x+4)=0$$

$$x+2=0 \text{ or } (x-1)^2=0 \text{ or } 5x+4=0$$

$$x=-2 \qquad x=1 \qquad x=\frac{-4}{5}$$

Interval	Test Point	y'	Interpretation
$x<-2$	-4	$+$	increasing
$-2<x<\frac{-4}{5}$	-1	$-$	decreasing
$\frac{-4}{5}<x<1$	0	$+$	increasing
$x>1$	3	$+$	increasing

3. Point A is the absolute minimum as it is the lowest point on the graph. Point C is a local minimum. This means that with reference to points in the immediate proximity of point C (neighbouring points), it is lower than the other points. However, with reference to the entire graph point C is not the lowest point.

   Point B is a local maximum with reference to its neighbouring points. There is no absolute maximum as the arrows indicate that the graph continues upward to infinity.

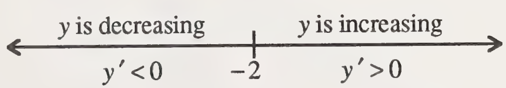
4. The point $(5,7)$ is an absolute maximum. The local maximum is at point $(-2,1)$. The point $(-5,-5)$ is an absolute minimum and $(0,-2)$ is a local minimum.

Activity 2

Define and determine the second derivative of a function, the points of inflection, and the characteristics of the turning points.

1. a.

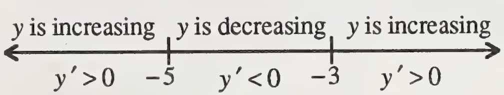
$$y = x^2 + 4x + 8$$
$$y' = 2x + 4 \qquad (\text{Let } y' = 0.)$$
$$2x + 4 = 0$$
$$5x = -2$$



b. $y = x^3 + 12x^2 + 45x + 16$

$$y' = 3x^2 + 24x + 45 = 0$$
$$3x^2 + 24x + 45 = 0$$
$$3(x^2 + 8x + 15) = 0$$
$$3(x+5)(x+3) = 0$$

$x = -5$ or $x = -3$



2. $f(x) = x^3 - 3x + 4$

$D_x y = 3x^2 - 3 = 0$ (Maximum and minimum values occur where $D_x y = 0$.)

$$3(x^2 - 1) = 0$$
$$x^2 - 1 = 0$$
$$x = \pm 1$$

At $x = 1$, $y = f(x) = 2$.

At $x = -1$, $y = f(x) = 6$.

$D_x^2 y = 6x$

At $x = 1$, $D_x^2 y = 6$. (Since this is positive, it is concave up; thus, 2 is a minimum.)

At $x = -1$, $D_x^2 y = -6$ (Since this is a negative, it is concave down; thus, 6 is maximum.)

Therefore, the local minimum value is 2 and the local maximum value is 6.

3. $y = x^3 - 3x^2 + 4x$

$y' = 3x^2 - 6x + 4$

At $y' = 0$, $3x^2 - 6x + 4 = 0$.

This does not factor. If you apply the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, you find that $b^2 - 4ac = 36 - 4(3)(4) = -12$.

Therefore, there are no real roots. There is no maximum or minimum for real values of x.

4. $y = x - 3 + \frac{4}{x}, x \neq 0$

$D_x y = 1 - 4x^{-2}$

Let $1 - 4x^{-2} = 0$.

$x^2 - 4 = 0$

$x = \pm 2$

$D_x^2 y = 8x^{-3}$

At $x = 2$, $y = 2 - 3 + \frac{4}{2} = 1$.

$D_x^2 y = 1$ (positive); thus, $(2, 1)$ is a minimum.

At $x = -2$, $y = (-2) - 3 + \frac{4}{(-2)} = -7$.

$D_x^2 y = -1$ (negative); thus, $(-1, -7)$ is a maximum.

Therefore, the local minimum value is greater than the local maximum value.

5. $y = x^3 + x^2 - 5x + 3$

$D_x y = 3x^2 + 2x - 5$

When y is a maximum or a minimum, $D_x y = 0$.

Let $3x^2 + 2x - 5 = 0$.

$(3x + 5)(x - 1) = 0$

$x = \frac{-5}{3}$ or $x = 1$

If $x = \frac{-5}{3}$, then $y = \frac{256}{27} = 9\frac{13}{27}$.

At $x = 1$, $y = 0$.

Therefore, the graph touches the x-axis at $x = 1$. The tangent point is $(1, 0)$.

6. $y = x + \frac{4}{x}, x \neq 0$

$D_x y = 1 - 4x^{-2}$

$D_x^2 y = 8x^{-3}$

$1 - 4x^{-2} = 0$ (Let $D_x y = 0$.)

$x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2$

At $x = 2$, $y = 4$ and $D_x^2 y = 1$ (positive); thus, this is a minimum.

At $x = -2, y = -4$ and $D_x^2 y = -1$ (negative); thus, this is a maximum.

Therefore, $y = 4$ is a minimum value and $y = -4$ is a maximum value.
When $x > 0, y > 0$.
When $x < 0, y < 0$.
When $y = 0$, x is not a real number.

There is no x-intercept. The graph does not pass through the x-axis. Since the minimum and maximum values of y are 4 and -4 respectively, y cannot have any values in the range $-4 < y < 4$.

7. a. $y = x^3 - 6x + 2$

$D_x y = 3x^2 - 6$

$D_x^2 y = 6x$

Let $6x = 0$.

$x = 0$

At $x = 0$, $y = 2$.

When $x > 0$, $D_x y$ is negative and $D_x^2 y$ is positive.

When $x < 0$, $D_x y$ is negative and $D_x^2 y$ is negative.

The slope stays negative through this point. There is no change of direction. Therefore, $(0, 2)$ is an inflection point. Also, the second derivative changes sign which is another check for the inflection point.

b. $y = 2x^4 - 8x^3$

$D_x y = 8x^3 - 24x^2$

$D_x^2 y = 24x^2 - 48x$

Let $24x^2 - 48x = 0$.

$24x(x-2) = 0$

$x = 0$ or $x = 2$

At $x = 0$, $y = 0$.

At $x = 2$, $y = -32$.

Therefore, $(0, 0)$ and $(2, -32)$ are the two inflection points since the slope stays negative on both sides of each point and the second derivative changes sign on both sides of each point.

c. $y = x(x+3)^{\frac{1}{2}}$

$$\begin{aligned} D_x y &= x\left(\frac{1}{2}\right)(x+3)^{-\frac{1}{2}} + (x+3)^{\frac{1}{2}} \\ &= (x+3)^{-\frac{1}{2}}\left[\frac{x}{2} + (x+3)\right] \\ &= (x+3)^{-\frac{1}{2}}\left(\frac{x+2x+6}{2}\right) \\ &= \frac{3x+6}{2(x+3)^{\frac{1}{2}}} \end{aligned}$$

$$\begin{aligned} D_x^2 y &= \frac{2(x+3)^{\frac{1}{3}}(3) - (3x+6)(2)\left(\frac{1}{2}\right)(x+3)^{-\frac{1}{2}}}{4(x+3)} \\ &= \frac{(x+3)^{-\frac{1}{2}}\left[6(x+3) - (3x+6)\right]}{4(x+3)} \\ &= \frac{3x+12}{4(x+3)^{\frac{3}{2}}} \end{aligned}$$

Let $D_x^2 y = 0$.

$3x + 12 = 0$

$x = -4$

When $x = -4$, $y = (-4)(-4+3)^{\frac{1}{2}}$. (imaginary number)

Therefore, there is no real point of inflection.

8. $y = 2x^3 - 15x^2 + 10x - 8$

$D_x y = 6x^2 - 30x + 10$

$D_x^2 y = 12x - 30$

Let $12x - 30 = 0$.

$x = 2.5$

If $x < 2.5$, then $D_x^2 y < 0$ (concave down).

If $x > 2.5$, then $D_x^2 y > 0$ (concave up).

The curve is concave up when x is greater than 2.5 and concave down when x is less than 2.5.

9. a. $y = x^4 - 2x^2 - 8$

$y' = 4x^3 - 4x$

$y'' = 12x^2 - 4$

Let $x^4 - 2x^2 - 8 = 0$.

$(x^2 - 4)(x^2 + 2) = 0$

$x^2 - 4 = 0$ or $x^2 + 2 = 0$ (In $x^2 + 2 = 0$, x is not a real number.)

$x = \pm 2$

Therefore, the x-intercepts are $(2, 0)$ and $(-2, 0)$.

Let $y' = 0$.

$4x^3 - 4x = 0$

$4x(x^2 - 1) = 0$

$x = 0$ or $x = \pm 1$

When $x = 0$, $y = -8$ and y'' is negative; thus, this is a maximum.

When $x = 1$, $y = -9$ and y'' is positive; thus, this is a minimum.

When $x = -1$, $y = -9$ and y'' is positive; thus, this is a minimum.

Let $12x^2 - 4 = 0$.

$$x^2 = \frac{4}{12}$$

$$= \frac{1}{3}$$

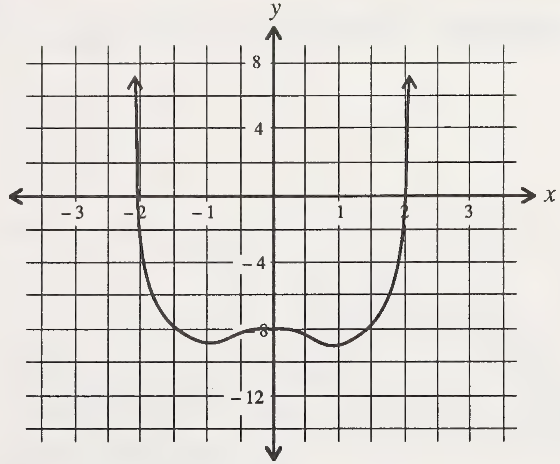
$$x = \pm\frac{\sqrt{3}}{3}$$

At $x = \pm\frac{\sqrt{3}}{3}$, $y = -\frac{77}{9}$.

When $x > \frac{\sqrt{3}}{3}$ or $x < \frac{-\sqrt{3}}{3}$, y'' is positive.

When $x < \frac{\sqrt{3}}{3}$ or $x > \frac{-\sqrt{3}}{3}$, y'' is negative.

Therefore, the inflection points are $\left(\frac{\sqrt{3}}{3}, -\frac{77}{9}\right)$ and $\left(-\frac{\sqrt{3}}{3}, -\frac{77}{9}\right)$.



b. $y=(x-2)^2(x+2)$

$y'=(x-2)(3x+2)$

$y''=6x-4$

Let $(x-2)^2(x+2)=0$.

$x=2$ or $x=-2$

Therefore, the x-intercepts are $(2, 0)$ and $(-2, 0)$,

Let $(x-2)(3x+2)=0$.

$x=2$ or $x=-\frac{2}{3}$

At, $x=2$, $y=0$ and $y''=8$.

Therefore, $(2, 0)$ is a minimum since y'' is positive.

At $x=-\frac{2}{3}$, $y=\frac{256}{27}$ and $y''=-8$.

Therefore, $\left(-\frac{2}{3}, \frac{256}{27}\right)$ is a maximum since y'' is negative.

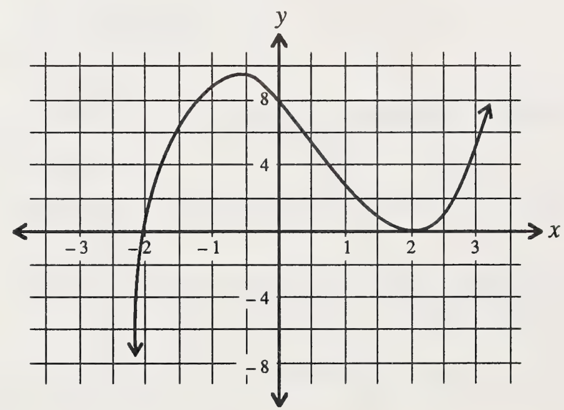
Let $6x-4=0$.

$x=\frac{4}{6}$

$=\frac{2}{3}$

At $x=\frac{2}{3}$, $y=\frac{128}{27}=4\frac{20}{27}$. y' is positive at $x>\frac{2}{3}$ or $x<\frac{2}{3}$.

Therefore, $\left(\frac{2}{3}, 4\frac{20}{27}\right)$ is an inflection point.



Activity 3

Determine the concavity, the position, and the direction of the graph of a polynomial function.

1. $y = x^3 - 2x^2 - 7x - 4$ ①

$= (x+1)(x+1)(x-4)$

$y' = 3x^2 - 4x - 7$ ②

$= (3x-7)(x+1)$

$y'' = 6x - 4$ ③

$= 2(3x-2)$

To find the x-intercept, let $y = 0$ in ①.
Therefore, $x = -1$, $x = -1$, and $x = 4$.

To find the y-intercept, let $x = 0$ in ①.
Therefore, $y = -4$.

To find the local maximum and minimum values, let $y' = 0$ in ②.

Therefore, $x = \frac{7}{3}$, and $x = -1$.

Substitute $x = \frac{7}{3}$ in ①.

$$y = \left(\frac{7}{3}\right)^3 - 2\left(\frac{7}{8}\right)^2 - 7\left(\frac{7}{3}\right) - 4$$
$$\doteq -18.52$$

The local minimum value is approximately -18.52 since y'' is positive.

Substitute $x = -1$ in ①.

$$y = (-1)^3 - 2(-1)^2 - 7(-1) - 4$$
$$= -1 - 2 + 7 - 4$$
$$= 0$$

The local maximum value is 0 since y'' is negative.

To determine the point of inflection, let $y'' = 0$ in ③.

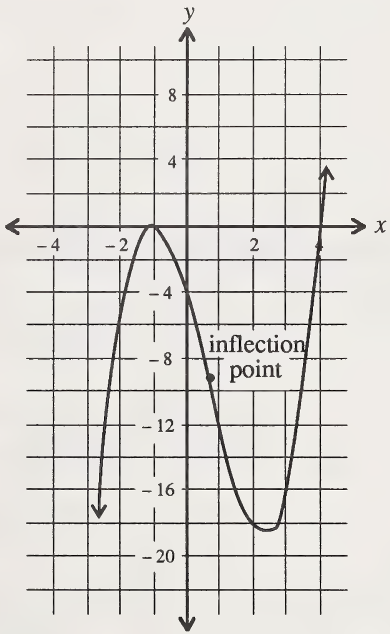
Therefore, $2(3x-2) = 0$; thus, $x = \frac{2}{3}$.

When $x = \frac{2}{3}, y \doteq -9.26$.

Test the neighbourhood of $x = \frac{2}{3}$ for continuity of the first derivative. The first derivative remains negative through this point.

A point of inflection exists at $\left(\frac{2}{3}, -9.26\right)$.

With this information you can sketch the curve and state the region of concavity.



When $y'=0$, a local maximum exist at $(-1,0)$ and a local minimum exists at $\left(2.\dot{3},-18.52\right)$.

When $y''=0$, a point of inflection exists at $\left(0.\dot{6},-9.26\right)$.

The function is concave down when $x<0.\dot{6}$ and concave up when $x>0.\dot{6}$.

2. $y=x^3-3x^2+4$

$y'=3x^2-6x=3x(x-2)$

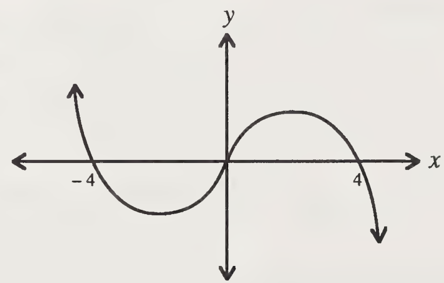
$y''=6x-6=6(x-1)$

Let $6x-6=0$; thus, $x=1$.

A point of inflection occurs at $(1,2)$ since the sign of the second derivative changes as the function passes through $(1,2)$.

A test for concavity shows that the curve is concave down for $x<1$ and concave up for $x>1$. When $x<1$, y'' is neagative; thus, the curve is concave down. When $x>1$, y'' is positive; thus, the curve is concave up.

3.

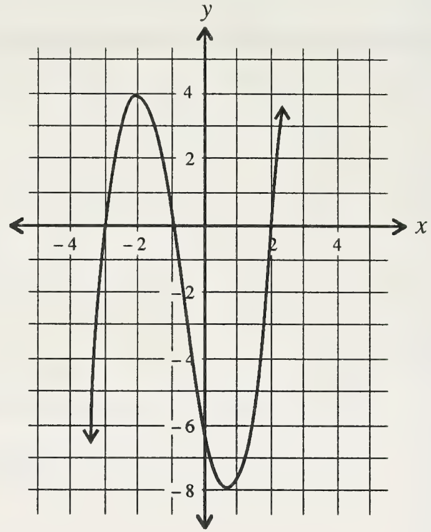


The function $f(x)$ is concave up for $x<0$ and concave down for $x>0$.

A point of inflection occurs at the origin $(0,0)$.

Extra Help

1. $y=(x-2)(x+1)(x+3)$ or $y=x^3+2x^2-5x-6$



The curve cuts the y-axis at $(0,-6)$ and cuts the x-axis at $(2,0)$, $(-1,0)$, and $(-3,0)$.

$$\frac{dy}{dx}=3x^2+4x-5=0$$

$x \doteq 0.8$ or $x \doteq -2.1$

Substitute $x=0.8$ in the original equation.

$$y=(0.8)^3+2(0.8)^2-5(0.8)-6$$
$$\doteq -8.2.$$

Substitute $x=-2.1$ in the original equation.

$$y=(-2.1)+2(-2.1)^2-5(-2.1)-6$$
$$\doteq 4.1$$

The turning points occur at approximately $(0.8,-8.2)$ and $(-2.1,4.1)$.

$\frac{dy}{dx}<0$ in the interval $-2.1<x<0.8$.

$\frac{dy}{dx}>0$ in the intervals $x<-2.1$ and $x>0.8$.

This is a cubic function with a positive controlling term. Therefore it will go up-down-up.

2. $f(x)=x^3+3x^2-24x+28$
   Let $x=0$.

$$y=(0)^3+3(0)^2-24(0)+28$$
$$=28$$

$$\frac{dy}{dx}=3x^2+6x-24$$

To find an extreme value, let $\frac{dy}{dx}=0$.

$$3x^2+6x-24=0$$
$$3\left(x^2+2x-8\right)=0$$
$$3(x-2)(x+4)=0$$

$x=2$ or $x=-4$

Substitute $x = 2$ in the original function.

$$y = (2)^3 + 3(2)^2 - 24(2) + 28$$
$$= 8 + 12 - 48 + 28$$
$$= 0$$

Substitute $x = -4$ in the original function.

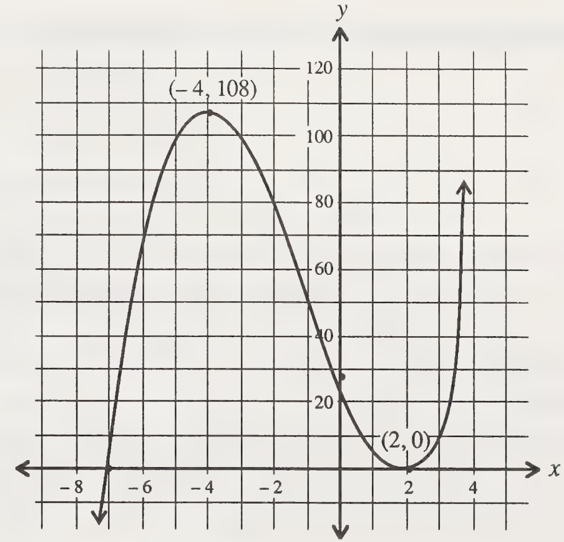
$$y = (-4)^3 + 3(-4)^2 - 24(-4) + 28$$
$$= -64 + 48 + 96 + 28$$
$$= 108$$

The turning points are at $(2, 0)$ and $(-4, 108)$.

To find the x-intercepts, let $y = 0$.

$$x^3 + 3x^2 - 24x + 28 = 0$$
$$(x-2)(x-2)(x+7) = 0$$
$$x = 2 \quad \text{or} \quad x = -7$$

Now draw the curve.



$\frac{dy}{dx} < 0$ in the interval $-4 < x < 2$.

$\frac{dy}{dx} > 0$ in the intervals $x < -4$ and $x > 2$.

Extensions

1. $\lim_{x\to 2}\left(\frac{3}{(x-2)^2}\right)$ is not defined.

   This suggests a vertical asymptote at $x = 2$.

   $$\lim_{x\to\infty}\left(\frac{3}{(x-2)^2}\right) = 0$$

   This suggests a horizontal asymptote at $y = 0$.

$$y = \frac{3}{(x-2)^2} = 3(x-2)^{-2}$$

$$D_x y = 3(-2)(x-2)^{-3}(1)$$

$$= \frac{-6}{(x-2)^3}$$

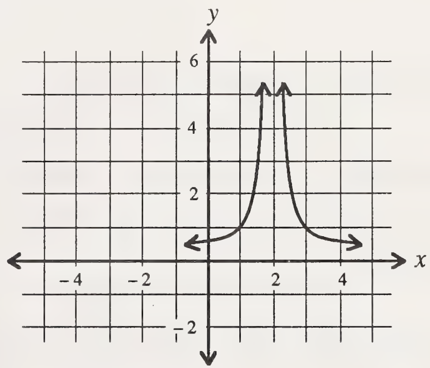
The equation $\frac{-6}{(x-2)^3} = 0$ has no solution. There are no turning points. $D_x y$ is not defined for $x = 2$. This confirms the vertical asymptote.

What is the interval of x over which $D_x y$ is positive?

When $(x-2)^3$ is negative, $x-2$ is negative and $\frac{-6}{(x-2)^3}$ is positive. Thus, $D_x y > 0$ when $x-2<0$ or $x<2$.

When $(x-2)^3$ is positive, $x-2$ is positive and $\frac{-6}{(x-2)^3}$ is negative. Thus, $D_x y < 0$ when $x-2>0$ or $x>2$.

The function is increasing when $x < 2$. (Slope is upward.)
The function is decreasing when $x > 2$. (Slope is downward.)



2. $$y^2 = 4x^3$$
$$D_x(y^2) = D_x(4x^3)$$
$$2y(D_x y) = 12x^2$$
$$D_x y = \frac{6x^2}{y}$$

Thus, $D_x y$ is not defined when $y = 0$. Therefore, the slope is not defined at point $(0, 0)$.

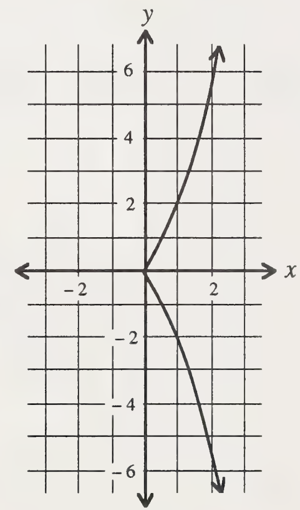
$D_x y = \frac{6x^2}{y}$ is positive when $y > 0$.

Thus, the curve slopes upward above the x-axis.

$D_x y = \frac{3x^2}{2y}$ is negative when $y < 0$. Thus, the curve slopes downward below the x-axis.

Therefore, there is a cusp at (0, 0).

The domain of x is the nonnegative real numbers because $y = \pm\sqrt{4x^3}$.



3. $$y = \frac{9x}{x^2 + 1}$$

$$D_x y = \frac{9(1 - x^2)}{(x^2 + 1)^2}$$

Let $\frac{9(1-x^2)}{(x^2+1)^2} = 0$; then solve for x to find the turning points.

To solve this equation, you must be secure in the knowledge of the properties of zero. Remember that the sides of an equation should not be multiplied or divided by zero.

Since $x^2 \neq -1$, you can multiply by $(x^2 + 1)^2$ and the equation becomes $9(1 - x^2) = 0$.

Divide both sides by 9 because $\frac{0}{9} = 0$.

This is equal to $1 - x^2 = 0$. Thus, $x = 1$ and $x = -1$.

The extreme values or turning points of $f(x)$ occur at $x = 1$ and $x = -1$.

When $|x| > 1$, $\frac{9(1-x^2)}{(x^2+1)^2}$ is negative.

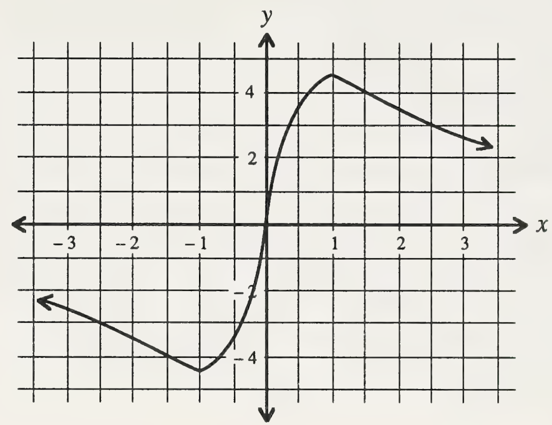
Thus, the curve has a negative slope in the intervals $x < -1$ and $x > 1$.

When $|x| < 1$, $\frac{9(1-x^2)}{(x^2+1)^2}$ is positive.

Thus, the curve has a positive slope in the interval $-1 < x < 1$.

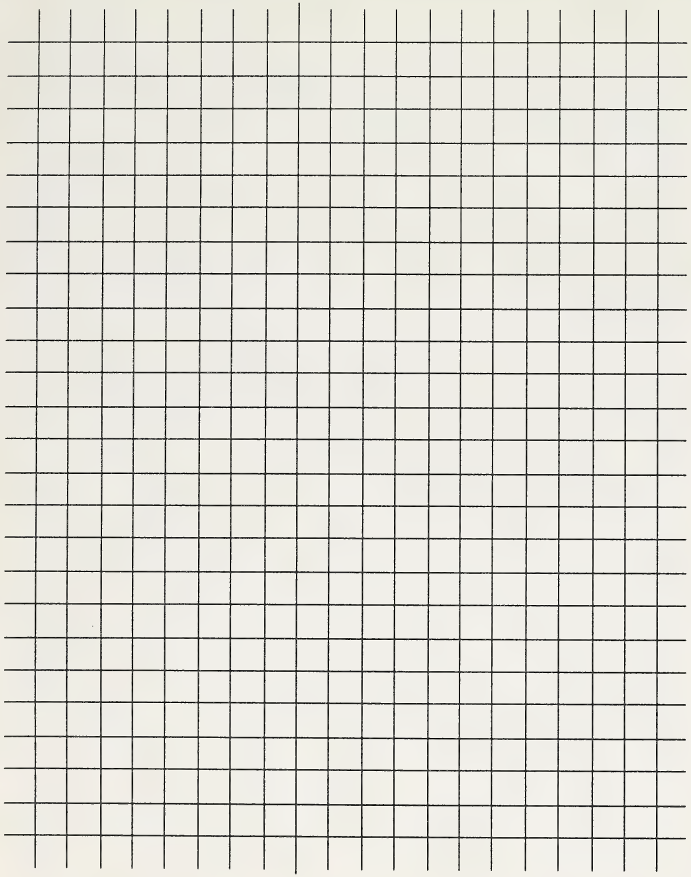
$$\lim_{x \to \infty} \left[\frac{9x}{x^2 + 1} \right] = 0$$

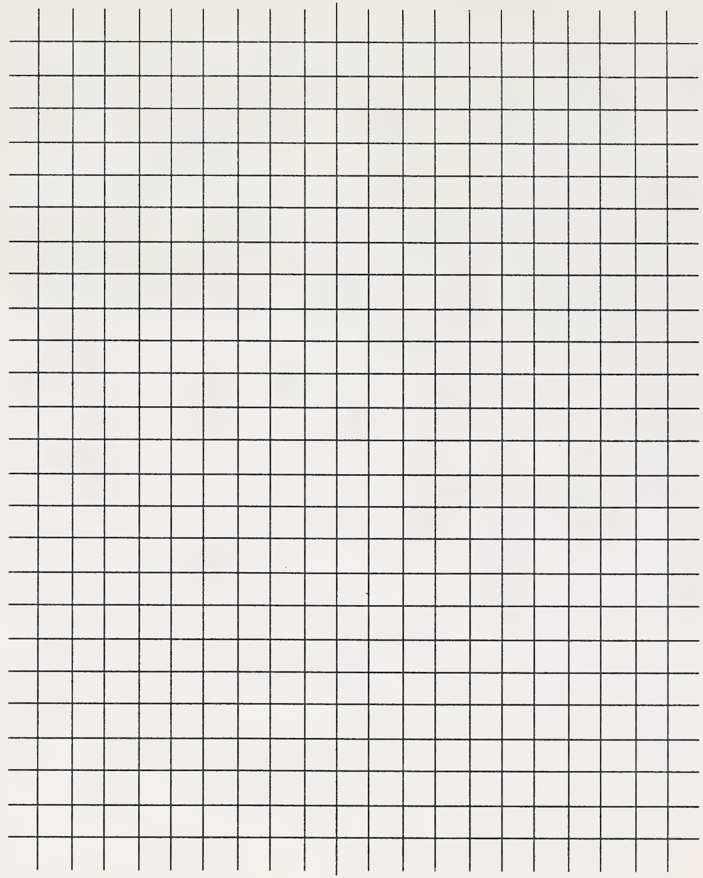
Therefore, $y = 0$ is a horizontal asymptote.



Appendix B
Graphing Material

Graph Paper





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